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The Cototal Hub Number in Graphs

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ABSTRACT

A set H of vertices in a graph G is a hub set of G, if for any $u, v \in V(G)H$, there is a uv −path with all intermediate vertices in H. A hub set H_{ct} is a cototal hub set of G if $G[V(G)H_{ct}]$ has no isolated vertex. The minimum cardinality of a cototal hub set is called the cototal hub number $h_{ct}(G)$ of G. In this paper, we study the cototal hub and interior cototal hub numbers of G. Also some results concerning of these parameters are established.

MSC: 05C40, 05C99.

Keywords: Cototal hub number, Interior cototal hub number, Hub number.

1. INTRODUCTION

Let $G = (V, E)$ be a finite and undirected graph without loops and multiple edges. A graph G is called (p, q) graph if G is with p vertices and q edges. And $\delta(G)(\Delta(G))$ denotes the minimum (maximum) degree among the vertices of G, respectively[2]. The difference between two sets A and B is denoted by $A \cup B$. For $v \in V(G)$, the open neighbourhood of v is denoted by $N(v) = \{u \in V(G): uv \in E(G)\}$, for $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$, similarly the closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$, and $N[S] = N(S) \cup S$. See[2] for terminology and notations not defined here.

Walsh[12] introduced the theory of hub in 2006, where a hub set H of G is a set of vertices in G such that any two vertices in $V(G) \cup H$ are connected by a path whose all internal vertices lie in H. The hub number of G, denoted by $h(G)$, is the minimum size of a hub set in G. A hub set H_r of G called a restrained hub set if for any two vertices outside H_r are connected by a path whose all internal vertices lie out H_r [6]. The contraction of a vertex x in G (denoted by G/x) as being the graph obtained by deleting x and putting a clique on the (open) neighbourhood of x, (note that this operation does not create multiple edges, if two neighbours of x are already adjacent, then they remain simply adjacent)[12]. For more details on the hub studies we refer to [4, 5, 7, 8, 9, 10, 11]. Graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively, their union, $G(V,E) = G_1 \cup G_2$ has as expected, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The corona $G \circ F$ of two graphs G and F is the graph obtained by taking one copy of G of order p and p copies of F, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of F. For every $v \in V(G)$, denote by F_v the copy of F whose vertices are attached one by one to the vertex v [1]. The following results will be useful in the proof of our results.

Theorem 1.1 [12] Let T be a tree with p vertices and l leaves. Then
$\mathrm{h}(\mathrm{G}) = \mathrm{h_c}(\mathrm{G}) = \mathrm{p} - \mathrm{l}$.

Theorem 1.2 [12] Let G be a graph with p vertices. Then

$$\mathrm{h}(\mathrm{G}) = \mathrm{p} - \Delta(\mathrm{G}).$$

Theorem 1.3 [3] If G is a connected graph. Then
$\mathrm{h}(\mathrm{G}) = \mathrm{h_c}(\mathrm{G}) = \gamma_c(\mathrm{G}) = \mathrm{h}(\mathrm{G}) + 1$.

Theorem 1.4 If G is a connected graph with p vertices. Then

$$\mathrm{h_r}(\mathrm{G}) + \mathrm{h_r}(\overline{\mathrm{G}}) - \mathrm{p}.$$

2. COTOTAL HUB NUMBER

We start our investigation with the following definition.

Definition 2.1 Let G be a graph. A hub set $\mathrm{H_{ct}}$ is a cototal hub set of G if $\mathrm{G}[\mathrm{V}(\mathrm{G}) = \mathrm{H_{ct}}]$ has no isolated vertex. The minimum cardinality of a cototal hub set is called the cototal hub number $\mathrm{h_{ct}}(\mathrm{G})$ of G.

It is easy to observe the following bound.

Proposition 2.1 For any graph G, $\mathrm{h}(\mathrm{G}) = \mathrm{h_{ct}}(\mathrm{G})$.

Theorem 2.1 Let T be a non trivial tree of order p. Then

$$\mathrm{h_{ct}}(\mathrm{T}) = \mathrm{p} - 2.$$

Proof. Let T be a non trivial tree of order p, and $\mathrm{H_{ct}}$ be a cototal hub set of T. Then we have to discuss the following cases:

Case 1: $\mathrm{T}[\mathrm{V}(\mathrm{T}) = \mathrm{H_{ct}}]$ is connected. Then $\mathrm{h_{ct}}(\mathrm{T}) = \mathrm{p} - 2$, if $|\mathrm{V}(\mathrm{T}) = \mathrm{H_{ct}}| = 3$, then $\mathrm{T}[\mathrm{V}(\mathrm{T}) = \mathrm{H_{ct}}] \cong \mathrm{P_k}, \mathrm{k} - 3$, and there is no path with all internal vertices in $\mathrm{H_{ct}}$ between the two vertices in the ends of the path $\mathrm{P_k}$.

Case 2: $\mathrm{T}[\mathrm{V}(\mathrm{T}) = \mathrm{H_{ct}}]$ is disconnected. Let $\mathrm{C_1}, \mathrm{C_2}$ be any two components of $\mathrm{T}[\mathrm{V}(\mathrm{T}) = \mathrm{H_{ct}}]$, then by definition of cototal hub set $|\mathrm{C_1}| = 2$, and $|\mathrm{C_2}| = 2$, let $\mathrm{v_1} \in \mathrm{V}(\mathrm{C_1})$ and $\mathrm{v_2} \in \mathrm{V}(\mathrm{C_2})$ such that $\mathrm{d}(\mathrm{v_1}, \mathrm{v_2}), \mathrm{d}(\mathrm{x}, \mathrm{y})$ for all $\mathrm{x} \in \mathrm{V}(\mathrm{C_1})$ and $\mathrm{y} \in \mathrm{V}(\mathrm{C_2})$, since any two vertices in a tree

has exactly one path between them. Therefore, the path between $z \in V(C_1)$ and $w \in V(C_2)$, includes x and y, hence H_{ct} is not a hub set of T, and that is a contradiction.
By two cases we get that only possible way to construct a cototal hub set of T, is through taking all vertices from T except any two adjacent vertices. Thus $h_{ct}(T) = p - 2$.

Proposition 2.2 For any graph G, $h_{ct}(G) = h_r(G) + 1$.

Proof. Let G be a graph, and H_r be a minimum restrained hub set of G, then by definition of H_r, $G[V(G) = H_r]$ is a connected graph. Hence H_r is a cototal hub set of G if $|V(G) = H_r| \neq 1$. Therefore, $h_{ct}(G), h_r(G) + 1$.

Proposition 2.3 Let G be a graph has at least one edge. Then $h_{ct}(G) = h_r(G)$.

Proof. Let G be a non trivial graph, and H_r be a minimum restrained hub set of G, by definition of restrained hub set, $G[V(G) = H_r]$ is connected, and since G has an edge, $|G[V(G) = H_r| = 2$. Therefore, H_{ct} is a cototal hub set, thus $h_{ct}(G) = h_r(G)$.

Corollary 2.1 Let G be a graph of order p, and has at least one edge. Then $h_{ct}(G) + h_{ct}(\overline{G}) = p$. And this bound is sharp.

Proof. Let G be a graph of order p, and has at least one edge, if G is complete graph, then the result holds. While if not, then by Proposition 2.3, and since $\overline{G}$ has an edge, we get that $h_{ct}(G) = h_r(G)$, and $h_{ct}(\overline{G}) + h_r(\overline{G})$, now by Theorem 1.4,

$$h_{ct}(G) + h_{ct}(\overline{G}) + h_r(G) + h_r(\overline{G}) = p.$$

For sharpness, let $G \cong P_4$. Then $h_{ct}(G) + h_{ct}(\overline{G}) = 2 + 2 = 4 = p$.

3. INTERIOR COTOTAL HUB NUMBER

We first introduce the concept of interior cototal hub number.
Definition 3.1 Let G be a graph with at most one isolated vertex. A hub set H_{ic} is an interior cototal hub set of G if $G[H_{ic}]$ has no isolated vertex. The minimum cardinality of an interior cototal hub set is called the interior cototal hub number $h_{ic}(G)$ of G.
Since any interior cototal hub set is a hub set. Then we get the following result.
Proposition 3.1 For any graph G, $h_{ic}(G) = h(G)$.

Proposition 3.2 Let G be a non trivial graph. Then
$h_{ic}(G) = h_{ct}(G) = 0$ if and only if G is complete graph.

Proposition 3.3 Let C_p be a cycle of order p, $p \neq 4$. Then

$$h_{ic}(C_p) = h_{ct}(G) = h(C_p) = p - 3.$$

Proof. Let C_p be a cycle of order $p \neq 4$, and let $v_1, v_2, .., v_p$ be a path in C_p. Take $H = \{v_1, v_2, \dots, v_{p-3}\}$, H is a cototal and interior cototal hub set of C_p, so $h_{ic}(C_p) = h_{ct}(G) = h(C_p) = p - 3$, but $h_{ic}(C_p)) = h(C_p) = p - 3$, also $h_{ct}(C_p) = h(C_p) = p - 3$. Therefore,

$h_{ic}(C_p) = h_{ct}(G) = h(C_p) = p - 3$.

Proposition 3.4 *Let T be a tree with p vertices and l leaves. Then*

$$h_{ic}(T) = \begin{cases} p - l, & if \ T \ncong K_{1,p-1}\ ; \\ 2, & if \ T \cong K_{1,p-1}\ . \end{cases}$$

Proof. Let T be a non trivial tree of order p. If $T \ncong K_{1,p-1}$, then H_{ic} the set of all vertices of the tree except its leaves, is an interior cototal hub set of T, so $h_{ic}(T) = p - l$, but $h_{ic}(T) = h(T) = p - l$ by Theorem 1.1. Therefore, $h_{ic}(T) = p - l$. While if $T \cong K_{1,p-1}$, then set contains the center and any leaf of $K_{1,p-1}$ is an interior cototal hub set of T, so $h_{ic}(T) = 2$. That concludes the proof.

Proposition 3.5 Let $G \cong K_{k_1,k_2,\ldots,k_n}$. Then

$$h_{ic}(G) = \begin{cases} 0, & \text{if } G \cong K_p\ ; \\ 2, & \text{if } G \ncong K_p\ . \end{cases}$$

Proof. Let $G \cong K_{k_1,k_2,\ldots,k_n}$ be a complete n −partite graph. If G is complete graph, then by Proposition 3.2, $h_{ic}(G) = 0$. While if not then $h_{ic}(G) - 2$, but $H_{ic} = \{x, y\}$, where x, y belongs to two different parts forms an interior cototal hub set. Therefore, $h_{ic} = 2$. And we are done.

Corollary 3.1 Let G, F be any two graphs. Then

$$h_{ic}(G + F) = \begin{cases} 0, & \text{if } G + F \cong K_p\ ; \\ 2, & \text{if } G + F \ncong K_p\ . \end{cases}$$

Proposition 3.6 Let G be a connected graph. Then

$$h(G) = h_{ic}(G) + h_c(G) + 1 + h(G) + 2.$$

Proof. Let G be a graph, and H_c be a minimum connected hub set of G, thus $G[H_c]$ is connected. If $|H_c| \neq 1$, then $G[H_c]$ has no isolated vertex. Thus, $h_{ic}(G) \cup h_c(G)$. While if $H_c = \{v\}$, then take $H_{ic} = H_c \cup \{u\}$, where u is adjacent to v, then H_{ic} is a minimum interior cototal hub set. Therefore, $h_{ic}(G) \cup h_c(G) + 1$. By Proposition 3.1 and Theorem 1.3 the result holds.

By previous proof the next corollary follows.
Corollary 3.2 Let G be a connected graph such that $h(G) \neq 1$. Then

$$h(G) \,\square\, h_{ic}(G) \,\square\, h_c(G) \,\square\, h(G) + 1.$$

Theorem 3.1 Let G be a graph such that $\Delta(G) \neq |V(F)| - 1$, where F is the component contained a vertex of maximum degree. Then $h_{ic}(G) \,\square\, p - \Delta(G)$.

Proof. Let G be a graph such that $\deg(v) = \Delta(G) \neq p - 1$. Let $H_{ic} = \{(N(v) \cup \{x\}) \square \{y\}\}$, where $y \in N(v)$ and $x \in N(y) \square N(v)$, this set is an interior cototal hub set, so $h_{ic}(G) \,\square\, p - \Delta(G)$.

Corollary 3.3 For any graph G. $h_{ic}(G) + h_{ic}(\overline{G}) \,\square\, p - \Delta(G) + 1$.

Proof. Let G be a graph. Assume that $H_{ic} = \{(N(v)\square\{x\})\}$, where $x \in N(v)$, this set is a cotoal hub set, so $h_{ic}(G) \square p - \Delta(G) + 1$.

Theorem 3.2 Let G be a graph such that $\Delta(G) \neq |V(F)| - 1$, where F is the component contained a vertex of maximum degree. Then

$$h_{ic}(G) + h_{ic}(\overline{G}) \square p + 1.$$

Proof. Let G be a non regular graph such that $h(G) \neq 1$. Then By Theorem 3.1,

$$\begin{aligned} h_{ic}(G) + h_{ic}(\overline{G}) &\square p - \Delta(G) + p - \Delta(\overline{G}) \\ &\square 2p - (\Delta(G) + (p - 1 - \delta(G))) \\ &\square p + 1 - (\Delta(G) - \delta(G)) \\ &\square p + 1. \end{aligned}$$

Corollary 3.4 Let G be a non regular graph such that $\Delta(G) \neq |V(F)| - 1$, where F is the component contained a vertex of maximum degree. Then

$$h_{ic}(G) + h_{ic}(\overline{G}) \square p.$$

And this bound is sharp.

Proof. Let G be a non regular graph such that $\Delta(G) \neq p - 1$. Since $\Delta(G) - \delta(G) \square 1$, we get that $h_{ic}(G) + h_{ic}(\overline{G}) \square p$, as in the previous proof.
The sharpness of the bound occurs when $G \cong P_4$

Theorem 3.3 Let G be a graph. Then $h_{ic}(G) = p - 1$ if and only if G has one isolated vertex, or the $|F| \square 3$ where F is a maximum component of G.

Proof. Let G be a graph, with one isolated vertex v, then the only interior cototal hub set of G is $V(G)\square\{v\}$, and if $|F| \square 3$ where F is a maximum component of G, then any interior cototal hub set of G must contain all vertices of G except some one vertex in some maximum component of G. Therefore in both cases $h_{ic}(G) = p - 1$. Conversely suppose that G has no isolated vertices and there is a maximum component F such that $|F| \square 4$. Then take $x, y \in V(F)$ such that $d(x, y) = d(F)$, the set $V(G)\square\{x, y\}$ is an interior cototal hub set of G, hence $h_{ic}(G) \square p - 2$, and this complete the proof.

Theorem 3.4 Let G, F be two graphs, where G is connected. Then

$$h_{ic}(G \circ F) = \begin{cases} 0, & \text{if } |V(G)| = 1, \quad \text{and } F \cong K_p\,; \\ 2, & \text{if } |V(G)| = 1, \quad \text{and } F \ncong K_p\,; \\ |V(G)|, & \text{if } |V(G)| \neq 1\,. \end{cases}$$

Proof. Let G, F be two graphs, where G is connected. Then we have to discuss the following cases:
Case 1: $|V(G)| = 1$ and $F \cong K_p$. Then $G \circ F$ is complete graph, hence by Proposition 3.2, $h_{ic}(G \circ F) = 0$.
Case 2: $|V(G)| = 1$ and $F \ncong K_p$. Then $G \circ F$ is not complete graph, hence $h_{ic}(G \circ F) \square 2$. Now, $V(G) \cup \{v\}$, where v is any vertex in F is an interior cototal hub set, so $h_{ic}(G \circ F) = 2$.

Case 3: $|V(G)| \neq 1$. Then $V(G)$ is an interior cototal hub set of $G \circ F$, suppose that there is a minimum interior cototal hub set with H_{ic}, such that $|H_{ic}| < |V(G)|$, then there is a vertex $x \in V(G) \setminus H_{ic}$ and $y \in F_x \setminus H_{ic}$. And so there is no path between y and any vertex in $V(G \circ F) \setminus (H_{ic} \cup F_x \cup \{x\})$, hence H_{ic} is not a hub set and not interior cototal hub set of $G \circ F$. Therefore, $h_{ic}(G \circ F) = |V(G)|$.

Proposition 3.7 Let G be a connected graph. If $h_c(G) \neq \gamma_c(G)$, then $h(G) = h_{ic}(G) = h_c(G)$.

Proof. Let G be a connected graph, and H_c be a minimum connected hub set of G, the assumption $h_c(G) \neq \gamma_c(G)$ implies that $N[H_c] \neq V(G)$. Let $v \in V(G) \setminus N[H_c]$, its clear that v adjacent to every vertex not in H_c, since H_c is a hub set, hence there is no isolated vertex in $G[V(G) \setminus N[H_c]]$. So H_c is an interior cototal hub set of G, and $h_{ic}(G) \leq h_c(G)$. Now by Theorem 1.3, we get that $h(G) \leq h_{ic}(G) \leq h_c(G) < \gamma_c(G) = h(G) + 1$. Therefore, $h(G) = h_{ic}(G) = h_c(G)$.

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