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Analysis and Application of Time Delay Model in Human Population Dynamics

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ABSTRACT

In this paper we present the time delay models to study the human population dynamics. The particular cases where there is discrete delay according to the sex involved and stage structure in the population growth were treated. The equilibrium and stability analysis of each of the cases were considered also. The stability analysis shows that the discrete delays in the population growth leads to instability in the growth.

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1. INTRODUCTION

Ordinary and partial differential equation have been used in population modelling for a long time. The Logistic model, Malthus model and Lotka-Volterra model are basics models and used to study the dynamical system. When we dealing with complexity in dynamical system and complicated phenomena then these models can be mislead. The delay differential equation is widely used in population dynamics and many other areas of applied sciences. Delay differential equations are popular tools for applied scientist to model their dynamical system. The concept of time delay plays an important role to study the human population dynamics and other dynamical framework. Time delays model are realistic models and based on the hypothesis that the rate of change of population size does not only depend of instant size but also depend on population size on earlier instant. The time delay models exhibits more complicated dynamics than the simple population models since a time delay can turn the stable equilibrium into unstable and fluctuate the population. They provide the rich framework for the analysis of dynamical process. Now a day many, time delay model are used to analysis the asymptotic and oscillatory behaviour of the dynamical system. Basically ordinary and partial differential equation based models required initial and boundary condition for the solution but time delay models depends on the solution at earlier time and initial history. In 1948, Hutchison modified Verhulst logistic model with time delay concept and presented a new delayed logistic equation as

$$\frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t-\tau)}{K}\right) \tag{1.1}$$

Where $\tau > 0$ is a time delay parameter, $P(t)$ is population size at time t, r is growth rate and $P(t-\tau)$ is Population at earlier time (t-τ), [3]. The rate of population change depends on time lag τ. This model have been extensively analyzed and investigated by many researcher and authors. The another modification of logistic equation with time delay becomes

$$\frac{dP(t)}{dt} = rP(t) - KP^2(t) - dP(t)\int_0^\infty P(t-\tau)w(\tau)d\tau \tag{1.2}$$

Here change of $\frac{dP}{dt}$ depends upon time lag $\tau \geq 0$ and weight function W(τ). But if there is no time delay then $t - \tau = 0$,[5]. A time delay model can be either continuous time delay or discrete time delay. Generally basic discrete delays equation can be written as

$$\frac{dP(t)}{dt} = f(t, P(t), P(t-\tau_1), ...P(t-\tau_n)) \tag{1.3}$$

Where $\tau_1 > \tau_2 > ...\tau_n \geq 0$ are delays[4,10]. In this paper we consider only discrete time delay models according to sex involved in population. The objectives of our work are to use delay population model in describing population growth and determine the stability in population with respect to change in age structure of different sex.

2. POPULATION GROWTH OF MALE USING TIME DELAY MODEL

(i) Time Delay Equation for Juvenile Population:

In this section developing a time delay equation for the human population which contain age structure without consider other details. Approximating, male population age structure into adult population $P_A(t)$ and juvenile population $P_j(t)$ and choosing age of 12 as the division line for the male population because the age of 12 is close to the fact that the male has become sexually mature so denoting this by $P_A(t-12)$. Juvenile are born in proportion to the current adult population and leave juvenile population dying or being adult. Here we neglect the migration effect in population. Hence the time delay equation fir juvenile is

$$\frac{dP_j(t)}{dt} = bP_A(t) - p_j bP_A(t-12) - d_j P_j(t) \tag{2.1}$$

Where b is birth rate (constant) and p_j is the surviving probability of juvenile to be adult.

To solve equation (2.1) using variable separation

$$\frac{dP_j(t)}{dt} + d_j P_j(t) = bP_A(t) - p_j bP_A(t-12) \tag{2.2}$$

Now multiplying equation (2.2) by an arbitrary function $\phi(t)$

$$\phi \frac{dP_j(t)}{dt} + \phi d_j P_j(t) = \phi[bP_A(t) - p_j bP_A(t-12)] \tag{2.3}$$

But $$\phi \frac{dP_j(t)}{dt} = \frac{d(\phi P_j(t))}{dt} - \frac{P_j(t) d\phi}{dt} \tag{2.4}$$

Using equation (2.4) in (2.3) we get

$$\frac{d(\phi P_j(t))}{dt} + \phi d_j P_j(t) - \frac{P_j(t) d\phi}{dt} = \phi[bP_A(t) - p_j bP_A(t-12)] \tag{2.5}$$

If $\left(\phi d_j - \frac{d\phi}{dt}\right) P_j(t) = 0$ where $P_j(t) \neq 0$ then $$\frac{d(\phi P_j(t))}{dt} = \phi[bP_A(t) - p_j bP_A(t-12)] \tag{2.6}$$

$\phi d_j - \frac{d\phi}{dt} = 0 \;\; \Rightarrow \frac{d\phi}{dt} = \phi d_j \Rightarrow \phi = e^{d_j \cdot t}$, this value of ϕ is the integrating factor.

Now integrating both side equation (2.6), then we have

$$\phi P_j(t) = \int \phi[bP_A(t) - p_j bP_A(t-12)] dt + c_1$$

$$P_j(t) = \phi^{-1} \int \phi[bP_A(t) - p_j bP_A(t-12)] dt + c_1$$

Let $bP_A(t) - p_j bP_A(t-12) = k$ then

$$P_j(t) = \phi^{-1} \int \phi k dt + c_1$$

$$P_j(t) = e^{-d_j t} \int k e^{d_j t} dt + c_1 \tag{2.7}$$

(ii) Time Delay Equation for Adult Population:

Considering the deaths are in proportion to present adults and death rate d_A is the proportional constant. Leaving the effect of migration and an adult can go out by dying. Then time delay equation for adult population is given by

$$\frac{dP_A(t)}{dt} = p_j bP_A(t-12) - d_A P_A(t) \tag{2.8}$$

For the solution of equation (2.7) collecting like terms

$$\frac{dP_A(t)}{dt} + d_A P_A(t) = p_j bP_A(t-12) \tag{2.9}$$

Multiplying by (2.9) by an arbitrary function $\phi(t)$

$$\phi\frac{dP_A(t)}{dt}+\phi d_A P_A(t)=p_j b P_A(t-12)\phi \tag{2.10}$$

But $\phi\frac{dP_A(t)}{dt}=\frac{d(\phi P_A(t))}{dt}-\frac{P_A(t)d\phi}{dt}$ (2.11)

Using equation (2.11) in equation (2.10) then

$$\frac{d(\phi P_A(t))}{dt}+\phi d_A P_A(t)-\frac{P_A(t)d\phi}{dt}=[p_j b P_A(t-12)]\phi \tag{2.12}$$

If $\left(\phi d_A-\frac{d\phi}{dt}\right)P_A(t)=0$ where $P_A(t)\neq 0$ then $\phi d_A-\frac{d\phi}{dt}=0 \Rightarrow \frac{d\phi}{dt}=\phi d_A \Rightarrow \phi=e^{d_A\cdot t}$ which is an integrating factor.

$$\frac{d(\phi P_A(t))}{dt}=[p_j b P_A(t-12)]\phi \tag{2.13}$$

By integrating both sides of equation (2.13) and then putting the value of ϕ

$$\phi P_A(t)=\int[p_j b P_A(t-12)]\phi dt+c_2 \tag{2.14}$$

$$P_A(t)=p_j b e^{-d_A t}\int e^{d_A t}P_A(t-12)dt+c_2 \tag{2.15}$$

3. POPULATION GROWTH OF FEMALE USING TIME DELAY MODEL

To define the female growth equation taking account the maturity behaviour of female and dividing the age structure into juvenile phase from age zero to 12, child bearing phase from age 12 to 45 and menopause from age 45 and above. The combination of child bearing and menopause class makes adult class (age 12 and above) for females.

(i) Time Delay Equation for Female Juvenile:

Assuming that change in population of female juvenile is in proportion to present child's growing age and juvenile can go out by becoming child bearing or by die. The time delay equation is

$$\frac{dP_{jf}(t)}{dt}=b_f C_f(t)-p_f b_f C_f(t-12)-d_{jf}P_{jf}(t) \tag{3.1}$$

Where,

P_{jf} = female juvenile, b_f = female birth rate, C_f = child bearing female, p_f = female surviving probability

Separating likes terms in equation (3.1)

$$\frac{dP_{jf}(t)}{dt}+d_{jf}P_{jf}(t)=b_f C_f(t)-p_f b_f C_f(t-12) \tag{3.2}$$

Multiplying by (3.2) by an arbitrary function $\varepsilon(t)$ then we have

$$\frac{\varepsilon dP_{jf}(t)}{dt}+\varepsilon d_{jf}P_{jf}(t)=\varepsilon[b_f C_f(t)-p_f b_f C_f(t-12)] \tag{3.3}$$

But $$\frac{\varepsilon dP_{jf}(t)}{dt}=\frac{d(\varepsilon P_{jf}(t))}{dt}-\frac{P_{jf}(t)d\varepsilon}{dt} \tag{3.4}$$

Using equation (3.4) in equation (3.3) we have

$$\frac{d(\varepsilon P_{jf}(t))}{dt}+\varepsilon d_{jf}P_{jf}(t)-\frac{P_{jf}(t)d\varepsilon}{dt}=\varepsilon[b_f C_f(t)-p_f b_f C_f(t-12)] \tag{3.5}$$

If $\left(\varepsilon d_{jf}-\frac{d\varepsilon}{dt}\right)P_{jf}(t)=0$ where $P_{jf}(t)\neq 0$ then

$$\frac{d(\varepsilon P_{jf}(t))}{dt}=\varepsilon[b_f C_f(t)-p_f b_f C_f(t-12)] \tag{3.6}$$

Taking, $\varepsilon d_{jf}-\frac{d\varepsilon}{dt}=0 \Rightarrow \varepsilon=e^{d_{jf}t}$. Now integrating equation (3.6) both sides then we have

$$P_{jf}(t)=\varepsilon^{-1}\left[\int \varepsilon[b_f C_f(t)-p_f b_f C_f(t-12)]dt+c_3\right] \tag{3.7}$$

Let $b_f C_f(t)-p_f b_f C_f(t-12)=x$ then

$$P_{jf}(t)=\varepsilon^{-1}\int \varepsilon x dt+c_3$$

$$P_{jf}(t)=e^{-d_{jf}t}\int e^{d_{jf}t}x dt+c_3 \tag{3.8}$$

(ii) Time Delay Equation for Child Bearing Female:

The change in childbearing population is in proportion to go out of childbearing female by to be menopause or dying. If M_f is menopause class and dcf is childbearing class's death rate then time delay equation is describe by

$$\frac{dC_f(t)}{dt}=p_f C_f(t-12)-p_f M_f(t-45)-d_{cf}C_f(t) \tag{3.9}$$

Separating like terms of equation (3.9) and then multiplying by an arbitrary function $\varepsilon(t)$, we have

$$\varepsilon\frac{dC_f(t)}{dt}+\varepsilon d_{cf}C_f(t)=\varepsilon[p_f C_f(t-12)-p_f M_f(t-45)] \tag{3.10}$$

If $\frac{\varepsilon dC_f(t)}{dt}=\frac{d(\varepsilon C_f(t))}{dt}-\frac{C_f(t)d\varepsilon}{dt}$ then equation (3.10) reduce to

$$\frac{d(\varepsilon C_f(t))}{dt}+\varepsilon d_{cf}C_f(t)-\frac{C_f(t)d\varepsilon}{dt}=\varepsilon[p_f C_f(t-12)-p_f M_f(t-45)] \tag{3.11}$$

If $\left(\varepsilon d_{cf}-\frac{d\varepsilon}{dt}\right)C_f(t)=0$ where $C_f(t)\neq 0$ then $\varepsilon d_{cf}-\frac{d\varepsilon}{dt}=0 \Rightarrow \varepsilon=e^{d_{cf}t}$

$$\frac{d(\varepsilon C_f(t))}{dt} = \varepsilon[p_f C_f(t-12) - p_f M_f(t-45)] \tag{3.12}$$

Integrating equation (3.12) both sides and then putting $\varepsilon = e^{d_{cf}t}$

$$C_f(t) = e^{-d_{cf}t} \int e^{d_{cf}t}[p_f C_f(t-12) - p_f M_f(t-45)]dt + c_4 \tag{3.13}$$

Let $p_f C_f(t-12) - p_f M_f(t-45) = y$ $\quad C_f(t) = e^{-d_{cf}t} \int y e^{d_{cf}t} dt + c_4$ (3.14)

(iii) Time Delay Equation for Menopause Female:

The change in the menopause class is in proportion to the present menopause females and they can leave the system only by dying. If d_{mf} denote the deaths rate in menopause class then time delay equation is

$$\frac{dM_f(t)}{dt} = p_f M_f(t-45) - d_{mf} M_f(t) \tag{3.15}$$

Separating like terms of equation (3.15) and then multiplying by an arbitrary function $\varepsilon(t)$, we have

$$\frac{\varepsilon dM_f(t)}{dt} + \varepsilon d_{mf} M_f(t) = \varepsilon[p_f M_f(t-45)] \tag{3.16}$$

But $\dfrac{\varepsilon dM_f(t)}{dt} = \dfrac{d(\varepsilon M_f(t))}{dt} - \dfrac{M_f(t)d\varepsilon}{dt}$ then equation (3.16) becomes

$$\frac{d(\varepsilon M_f(t))}{dt} + \varepsilon d_{mf} M_f(t) - \frac{M_f(t)d\varepsilon}{dt} = \varepsilon[p_f M_f(t-45)] \tag{3.17}$$

$\left(\varepsilon d_{mf} - \dfrac{d\varepsilon}{dt}\right) M_f(t) = 0$ where $\mathrm{M}_f(t) \neq 0$ Then $\varepsilon d_{mf} - \dfrac{d\varepsilon}{dt} = 0 \ \Rightarrow \varepsilon = e^{d_{mf}t}$

$$\frac{d(\varepsilon M_f(t))}{dt} = \varepsilon[p_f M_f(t-45)] \tag{3.18}$$

Integrating equation (3.18) both sides and then putting $\varepsilon = e^{d_{mf}t}$. We have

$$M_f(t) = \varepsilon^{-1} \int \varepsilon[p_f M_f(t-45)]dt + c_5$$

$$M_f(t) = p_f e^{-d_{mf}t} \int e^{d_{mf}t}[M_f(t-45)]dt + c_5 \tag{3.19}$$

4. ANALYSIS OF TIME DELAY MODEL EQUATIONS

For the estimation of equilibrium states of the time delay model setting all delay differential equations equal to zero. Then we have

$$\frac{dP_j(t)}{dt} = bP_A(t) - p_j bP_A(t-12) - d_j P_j(t) = 0 \tag{4.1}$$

Solving equation (4.1), we find

$$P_j(t) = \frac{bP_A(t) - p_j bP_A(t-12)}{d_j} \tag{4.2}$$

Equation (4.2) shows that the rate of juvenile is proportional to the current adult population with the leaving of the juvenile to become adult and inversely proportional to the death of juvenile.

$$\frac{dP_A(t)}{dt} = p_j bP_A(t-12) - d_A P_A(t) = 0 \tag{4.3}$$

$$P_A(t) = \frac{p_j bP_A(t-12)}{d_A} \tag{4.4}$$

Equation (4.4) implies that the rate of adult population is proportional to adult population of leaving juvenile and inversely to the death of adult.

$$\frac{dP_{jf}(t)}{dt} = b_f C_f(t) - p_f b_f C_f(t-12) - d_{jf} P_{jf}(t) = 0 \tag{4.5}$$

$$P_{jf}(t) = \frac{b_f C_f(t) - p_f b_f C_f(t-12)}{d_{jf}} \tag{4.6}$$

In equation (4.6), the rate of juvenile female is proportional to child bearing female and the leaving juvenile to become child bearing age and inversely to the death of juvenile female.

$$\frac{dC_f(t)}{dt} = p_f C_f(t-12) - p_f M_f(t-45) - d_{cf} C_f(t) = 0 \tag{4.7}$$

$$C_f(t) = \frac{p_f C_f(t-12) - p_f M_f(t-45)}{d_{cf}} \tag{4.8}$$

Equation (4.8) declares that the rate of child bearing female is proportional to the leaving of child bearing female and by being menopause and inversely to the death of child bearing female.

$$\frac{dM_f(t)}{dt} = p_f M_f(t-45) - d_{mf} M_f(t) = 0 \tag{4.9}$$

$$M_f(t) = \frac{p_f M_f(t-45)}{d_{mf}} \tag{4.10}$$

Equation (4.10) shows that the rate at which the menopause female is proportional to becoming menopause and inversely to death of menopause female.

Furthermore, for the analysis of stability of above time delay models, we are setting a jacobian matrix of the delay equations for male population.

$$J(P_j, P_A) = \begin{pmatrix} \frac{\partial u}{\partial P_j} & \frac{\partial u}{\partial P_A} \\ \frac{\partial v}{\partial P_j} & \frac{\partial v}{\partial P_A} \end{pmatrix}, \text{ where } \begin{aligned} u &= bP_A(t) - p_j bP_A(t-12) - d_j P_j(t) \\ v &= p_j bP_A(t-12) - d_A P_A(t) \end{aligned}$$

If $\frac{\partial u}{\partial P_j} = -d_j, \frac{\partial u}{\partial P_A} = b, \frac{\partial v}{\partial P_j} = 0, \frac{\partial v}{\partial P_A} = -d_A$, then $J(P_j, P_A) = \begin{pmatrix} -d_j & b \\ 0 & -d_A \end{pmatrix}$.

For the evaluation of stable position using Eigen values

$$J(P_j, P_A) = \begin{pmatrix} -d_j & b \\ 0 & -d_A \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0 \quad \Rightarrow \begin{vmatrix} -d_j - \lambda & b \\ 0 & -d_A - \lambda \end{vmatrix} = 0$$

$$(-d_j - \lambda)(-d_A - \lambda) = 0 \Rightarrow -d_j - \lambda = 0 \text{ or } -d_A - \lambda = 0, \quad \text{then } \lambda_1 = d_j \text{ and } \lambda_2 = d_A$$

Here λ_1 = juvenile deaths and λ_2 = adult deaths. Due to short term time lag $d_A > d_j$. It consequently, implies that the time delay does not exceed the dominant time scale. During this the system is unstable and solely be stable if juvenile deaths in regards to time delay is equal to adult death in anticipated survival time.

$\Rightarrow \quad p_j > d_A - d_j$ And we considered the condition $d_A - d_j = 0 \quad \Rightarrow d_A = d_j$.

Next we analysing the stability situation of time delays model for females. For this, setting a Jacobian matrix

$$J(P_{jf}, C_f, M_f) = \begin{pmatrix} \frac{\partial u}{\partial P_{jf}} & \frac{\partial u}{\partial C_f} & \frac{\partial u}{\partial M_f} \\ \frac{\partial v}{\partial P_{jf}} & \frac{\partial v}{\partial C_f} & \frac{\partial v}{\partial M_f} \\ \frac{\partial w}{\partial P_{jf}} & \frac{\partial w}{\partial C_f} & \frac{\partial w}{\partial M_f} \end{pmatrix} \quad \text{Where} \quad \begin{aligned} u &= b_f C_f(t) - p_f b_f C_f(t-12) - d_{jf} P_{jf}(t) \\ v &= p_f C_f(t-12) - p_f M_f(t-45) - d_{cf} C_f(t) \\ w &= p_f M_f(t-45) - d_{mf} M_f(t) \end{aligned}$$

And

$$\frac{\partial u}{\partial P_{jf}} = -d_{jf}, \frac{\partial u}{\partial C_f} = b_f, \frac{\partial u}{\partial M_f} = 0, \ \frac{\partial v}{\partial P_{jf}} = 0, \frac{\partial v}{\partial C_f} = -d_{cf}, \frac{\partial v}{\partial M_f} = 0, \frac{\partial w}{\partial P_{jf}} = 0, \frac{\partial w}{\partial C_f} = 0, \frac{\partial w}{\partial M_f} = -d_{mf}$$

Now putting all values of derivatives in matrix, we have $J(P_{jf}, C_f, M_f) = \begin{pmatrix} -d_{jf} & b_f & 0 \\ 0 & -d_{cf} & 0 \\ 0 & 0 & -d_{mf} \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -d_{jf} & b_f & 0 \\ 0 & -d_{cf} & 0 \\ 0 & 0 & -d_{mf} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0 \quad \Rightarrow \begin{vmatrix} -d_{jf} - \lambda & b_f & 0 \\ 0 & -d_{cf} - \lambda & 0 \\ 0 & 0 & -d_{mf} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (-d_{jf} - \lambda)(-d_{cf} - \lambda)(-d_{mf} - \lambda) = 0 \quad \Rightarrow \quad \lambda_1 = d_{jf}, \lambda_2 = d_{cf}, \lambda_3 = d_{mf}$$

For the stability criteria in of system, the population death cannot be negative and all the Eigen values should be non negative. Therefore from the above illustrations system showing the instability.

5. CONCLUSION

The concept of time delay in population growth model plays a significant role to understand the structure and behaviour of dynamical system through initial history of the system. In the delay of population growth model, we develop a model for human being which incorporates age structure into the population. The models are subdivided into male delayed and female delayed differential equation model. For the formulation of time delay model, we are able to include the dependency of birth and mortality in the form of age without using partial differential equation. For the male population the age structure was divided into adult and juvenile whereby the juvenile delayed for 12 years before maturing to adult for reproduction. Similarly female population also divided into three categories, juvenile, child bearing female and menopause female. Also $P_A(t-12)$, which is current adult population that is beyond 12 years, will become the maturation period of the juvenile female. Adult can also leave it by age or dying which can also take time to regeneration in other organism. The time delay equation of male juvenile shows the changes between the becoming and leaving of juvenile respectively, while equation of adults indicate to be adult and leaving adult as well. And solution of these equations indicates that there is exponentially growth in juvenile and adult population as time increase. But the system tends to exponential decrease. The stability analysis of all time delays equation indicates the instability in the system.

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