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Some Types of RICCI Solitons on Para-Sasakian Manifolds

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ABSTRACT

The present paper deals with the study of Ricci solitons on para-Sasakian manifolds with generalized quasi conformal curvature tensor.

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Keywords: Para-Sasakian manifold, generalized quasi conformal curvature tensor, Ricci soliton, semisymmetric and Ricci-semisymmetric.

1. INTRODUCTION

In 1968, Yano and Sawaki[23] introduced the notion of quasi-conformal curvature tensor in the context of Riemannian geometry. Recently, Baishya and Chowdhury[3] studied generalized quasi-conformal curvature tensor in terms of $N(k,\mu)$-contact metric manifold. The important feature of generalized quasi-conformal curvature tensor lies in the fact that it consists of Riemannian curvature tensor R, conformal curvature tensor C[8], conharmonic curvature tensor $\hat{C}$[13], concircular curvature tensor E[22], projective curvature tensor P[22] and m-projective curvature tensor H[17] as particular cases. The generalized quasi-conformal curvature tensor W for an n-dimension Riemannian manifold $(M,\ g)$ is given by

$$\begin{aligned} W(X,Y)Z = \frac{(n-1)}{n}&[\{1+(n-1)a-b\}-\{1+(n-1)(a+b)\}c]C(X,Y)Z \\ &+[1-b+(n-1)a]E(X,Y)Z+(n-1)(b-a)P(X,Y)Z \\ &+\tfrac{(n-1)}{n}(c-1)\{1+(n-1)(a+b)\}\hat{C}(X,Y)Z, \end{aligned} \tag{1.1}$$

for all vector fields $X,\ Y,\ Z\ \in \chi(M)$, where the scalars a, b and c are real constants.
The equation (1.1) can also be written as

$$W(X,Y)Z = R(X,Y)Z + a[S(Y,Z)X - S(X,Z)Y] + b[g(Y,Z)QX - g(X,Z)QY]$$

$$-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(Y,Z)X-g(X,Z)Y] \tag{1.2}$$

where R, S, Q and r being Riemannian curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively.

Hamilton[9] introduced and studied the notion of Ricci flow which was a powerful tool to understand the manifolds with positive curvature. The Ricci flow on a Riemannian manifold is defined as

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij} \tag{1.3}$$

A Ricci soliton is a solution of the Ricci flow which moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of Einstein metric such that[10]

$$\mathcal{L}_V g + 2S + 2\lambda g = 0\,, \tag{1.4}$$

where S is the Ricci tensor and $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding depending on λ is negative, zero, and positive respectively. Many mathematicians drew their attention towards the geometry of Ricci soliton in past two decades. It has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. Sharma[20] initiated the study of the Ricci solitons in contact geometry. Thereafter, Ricci solitons in contact metric manifolds have been studied by several authors such as Bagewadi *et al.*[2], Bejan and Crasmareanu[4], Hui *et al.*[5,12,21], Deshmukh *et al.*[7], De *et al.*[6], He and Zhu[11], Tripathi[16], Baishya[14], Prakash *et al.*[18] and many others.

Motivated by the above studies, in this paper we study Ricci soliton on para-Sasakian manifold. The paper is constructed as follows: In Section 2, we present some basic notions of para-Sasakian manifold and Ricci soliton on it. Section 3 is devoted to study Ricci soliton (g, ξ, λ) on a generalized quasi conformally flat para-Sasakian manifold and we found the conditions for Ricci soliton to be steady, shrinking and expanding. Section 4 is concerned with generalized quasi conformally semi-symmetric para-Sasakian manifold admitting a Ricci soliton (g, V, λ) which is shrinking for $a \le b$ and expanding for $a > b\frac{n}{n-1}$. Finally in section 5, we have pointed out that the Ricci soliton (g, ξ, λ) in a para-Sasakian manifold satisfying $W \cdot S = 0$ is always expanding.

2. PRELIMINARIES

A differential manifold M is said to admit an almost paracontact Riemannian structure (ϕ, ξ, η, g), where ϕ is a (1,1)-tensor field, ξ is a characteristic vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi 2 = I - \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \tag{2.1}$$

$$g(X,\xi) = \eta(X),\ \ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields $X, Y \in T(M)$. If (ϕ, ξ, η, g) on M satisfies the following equations

$$(\nabla_X \phi) = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

$$d\eta = 0 \text{ and } \nabla_X \xi = \phi X \tag{2.4}$$

then M is called para-Sasakian manifold[1].

In a para-Sasakian manifold, the following relations hold[1, 15, 19]:

$$(\nabla_X \eta)Y = -g(X,Y) - \eta(X)\eta(Y), \tag{2.5}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.7}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.9}$$

Let (g, V, λ) be a Ricci soliton in an n-dimensional para-Sasakian manifold M.

From (2.4), we have

$$\mathcal{L}_\xi g(X,Y) = 2g(\phi X, Y) \tag{2.10}$$

Also from (1.4) and (2.10), we find that

$$S(X,Y) = -\lambda g(X,Y) - g(\phi X, Y) \tag{2.11}$$

3. RICCI SOLITON IN A GENERALIZED QUASI CONFORMALLY FLAT PARA-SASAKIAN MANIFOLD

Let us consider a generalized quasi conformally flat para-Sasakian manifold $(M,\ g)$. Then from (1.2), we have

$$R(X,Y)Z = a[S(X,Z)Y - S(Y,Z)X] + b[g(X,Z)QY - g(Y,Z)QX] - \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)[g(X,Z)Y - g(Y,Z)X] \tag{3.1}$$

Taking inner product of (3.1) with W, we get

$$R(X,Y,Z,W) = a[S(X,Z)g(Y,W) - S(Y,Z)g(X,W)]$$
$$+b[g(X,Z)S(Y,W) - g(Y,Z)S(X,W)]$$
$$-\frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \tag{3.2}$$

Putting $X = W = e_i$ in (3.2) and summing over $i = 1, 2, \ldots\ldots\ldots\ldots, n$ in (3.2), we obtain

$$S(Y,Z) = \left[\frac{br(c-1)}{1+a(n-1)-b} + \frac{cr}{n}\right] g(Y,Z) \tag{3.3}$$

Using (2.11) in (3.3), we get

$$\left\{\lambda + r\left[\frac{b(c-1)}{1+a(n-1)-b} + \frac{c}{n}\right]\right\} g(Y,Z) = -g(\phi Y, Z) \tag{3.4}$$

Taking $Z = \xi$ in (3.4), we get

$$\lambda = -r\left[\frac{b(c-1)}{1+a(n-1)-b} + \frac{c}{n}\right] \tag{3.5}$$

Therefore we can state the following:

Theorem 3.1. A Ricci soliton (g, ξ, λ) on a generalized quasi conformally flat para-Sasakian manifold M is steady for $b = 0$ and $c = 0$ or zero scalar curvature, shrinking for $c = 1$ and positive scalar curvature and expanding for $c = 1$ and negative scalar curvature.

4. RICCI SOLITON IN A GENERALIZED QUASI CONFORMALLY SEMI-SYMMETRIC PARA-SASAKIAN MANIFOLD

A para-Sasakian manifold M is said to be semi-symmetric with respect to generalized quasi conformal curvature tensor if

$$(R(\xi,X)\cdot W)(\xi,Z)U=0 \tag{4.1}$$

Equation (4.1) can be written as

$$R(\xi,X)W(\xi,Z)U-W(R(\xi,X)\xi,Z)U-W(\xi,R(\xi,X)Z)U-W(\xi,Z)R(\xi,X)=0 \tag{4.2}$$

By using (2.7) and (1.2), the above equation becomes

$$\begin{aligned}&\{\eta(U)\eta(Z)-g(U,Z)+a[S(Z,U)+(n-1)\eta(U)\eta(Z)]-b[(n-1)g(Z,U)\\&+\eta(U)\eta(Z)]-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(Z,U)-\eta(U)\eta(Z)]\}X-\{\eta(U)g(X,Z)\\&-\eta(X)g(U,Z)+a[S(Z,U)\eta(X)+(n-1)\eta(U)g(X,Z)]-b[(n-1)g(Z,U)\eta(X)\\&+\eta(U)g(X,Z)]-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(Z,U)\eta(X)-\eta(U)g(X,Z)]\}\xi\\&-W(X,Z)U+\eta(X)\{\eta(U)Z-g(U,Z)\xi+a[S(Z,U)\xi+(n-1)\eta(U)Z]\\&-b[(n-1)g(Z,U)\xi+\eta(U)Z]-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(Z,U)\xi+\eta(U)Z]\}\\&-\eta(Z)\{\eta(U)X-g(U,X)\xi+a[S(X,U)\xi+(n-1)\eta(U)X]\\&-b[(n-1)g(X,U)\xi+\eta(U)X]-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(X,U)\xi-\eta(U)X]\}\\&-\eta(U)\{\eta(X)Z-g(X,Z)\xi+\ a[S(X,Z)\xi+(n-1)\eta(X)Z]\\&-b[(n-1)g(X,Z)\xi+\eta(X)Z]-\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)[g(X,Z)\xi-\eta(X)Z]\}\\&+g(X,U)\left\{\left[1+(a+b)(n-1)+\frac{cr}{n}\left(\frac{1}{n-1}+a+b\right)\right](Z-\eta(Z)\xi)\right\}=0\end{aligned} \tag{4.3}$$

Taking inner product with ξ and using (1.2) and (2.6), we obtained

$$S(X,Z)=\left[\left(\frac{b}{a}-1\right)n+1\right]g(X,Z). \tag{4.4}$$

Using (2.11) in (4.4), we get

$$\left[\lambda+\left(\frac{b}{a}-1\right)n+1\right]g(X,Z)=-g(\phi X,Z) \tag{4.5}$$

Putting $Z\ =\ \xi$ in (4.5), we get

$$\lambda=\left(1-\frac{b}{a}\right)n-1 \tag{4.6}$$

Thus we have the following theorem:

Theorem 4.2. A Ricci soliton (g,ξ,λ) on a generalized quasi conformally semi-symmmetric para-Sasakian manifold M is shrinking for $a\le b$ and expanding for $a>b\left(\frac{n}{n-1}\right)$.

Further let (g,V,λ) be Ricci soliton in a para-Sasakian manifold (M,g). If V is generalized quasi conformally killing vector field, then by definition

$$\mathcal{L}_V g=\rho g, \tag{4.7}$$

for some scalar function ρ. From (1.4) and (4.7), we have

$$S(X,Y)=-(\lambda+\frac{\rho}{2})g(X,Y). \tag{4.8}$$

From (4.2), we have

$$\begin{aligned}(R(\xi,X)\cdot W)(Y,Z)=&R(\xi,X)W(Y,Z)U-W(R(\xi,X)Y,Z)U\\&-W(Y,R(\xi,X)Z)U-W(Y,Z)R(\xi,X)U\end{aligned} \tag{4.9}$$

By using (2.6), (2.7), (2.8) and (4.8) in (4.9), one can obtain

$$R\cdot W=0 \tag{4.10}$$

This leads to the following:

Theorem 4.3. An n-dimensional para-Sasakian manifold (M, g) admitting a Ricci soliton (g, V, λ) is generalized quasi conformally semi-symmetric, if V is a confomally killing vector field.

5. RICCI SOLITON IN PARA-SASAKIAN MANIFOLD SATISFYING $W \cdot S = 0$

Let (M, g) be a para-Sasakian manifold satisfying the condition $W(\xi, X) \cdot S = 0$. This can also be written as

$$S(W(\xi, X)Y, Z) + S(Y, W(\xi, X)Z) = 0 \tag{5.1}$$

Putting $Z = \xi$ in (5.1) and using (2.8), we get

$$-(n-1)\eta(W(\xi, X)Y) + S(Y, W(\xi, X)\xi = 0 \tag{5.2}$$

By virtue of (1:1) and then using (2.6), we obtain

$$\eta(W(\xi, X)Y) = aS(X, Y) + \left[1 + (a+b)(n-1) + \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right]\eta(X)\eta(Y) - \left[1 + b(n-1) + \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right] g(X, Y) \tag{5.3}$$

$$S(Y, W(\xi, X)\xi) = \left[1 + (a+b)(n-1) + \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right] S(X, Y) + \left[1 + (a+b)(n-1) + \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right](n-1)\eta(X)\eta(Y) \tag{5.4}$$

Substituting (5.3) and (5.4) in (5.2), we get

$$\left[1 + b(n-1) + \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right][S(X, Y) + (n-1)g(X, Y)] = 0. \tag{5.5}$$

Using (2.11) and then by taking $Y = \xi$ in (5.5), we obtain

$$(n-1-\lambda)\left[1 + b(n-1) + \frac{cr}{n}\left(\frac{1}{n-1} + a + b\right)\right]\eta(X) = 0. \tag{5.6}$$

From equation (5.6), one can state the following:

Theorem 5.4. Ricci soliton (g, ξ, λ) in a para-Sasakian manifold satisfying $W \cdot S = 0$ is expanding provided $r \neq \frac{-n(n-1)[1+b(n-1)]}{c[1+(a+b)(n-1)]}$.

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