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The Forcing Open Geodetic Domination Number of a Graph

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ABSTRACT

Let G be a connected graph with at least two vertices and S a minimum open geodetic dominating set of G. A subset T of S is called a forcing subset for S if S is the unique minimum open geodetic dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing open geodetic dominating number of S, denoted by $f_{\gamma og}(S)$, is the cardinality of a minimum forcing subset of S. The forcing open geodetic dominating set of G denoted by $f_{\gamma og}(G)$ is $f_{\gamma og}(G)$ = min{$f_{\gamma og}(S)$}, where the minimum is taken over all minimum forcing open geodetic dominating sets in G. Some general properties satisfied by this concept are studied. For every pair a, b of integers with $0 \leq a < b$ and $b \geq 6$, there exists a connected graph G such that $f_{\gamma og}$(G) = a and $\gamma_{og}(G)$ = b.

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Keywords: open geodetic dominating number, forcing open geodetic dominating number.

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to Harary[10]. The *distance* $d(u, v)$ between two vertices

u and v in a connected graph G is the length of a shortest u–v path in G. An u–v path of length $d(u,v)$ is called an u–v *geodesic*. A vertex x is said to lie on a $u-v$ geodesic P if x is a vertex of P including the vertices u and v. The closed interval $I[x,y]$ consists of x,y and all vertices lying on some $x-y$ geodesic of G. For a non-empty set $S \subseteq V(G)$, the set $I[S] = \bigcup_{x,y\in S} I[x,y]$ is the *closure* of S. A set $S \subseteq V(G)$ is called a *geodetic* set if $I[S] = V(G)$.Thus every vertex of G is contained in a geodesic joining some pair of vertices in S. The minimum cardinality of a geodetic set of G is called the *geodetic number* of G and is denoted by $g(G)$. A geodetic set of minimum cardinality is called g-*set* of G. $N(v) = \{u \in V(G): uv \in E(G)\}$ is called the *neighborhood* of the vertex v in G. A vertex v is a *extreme vertex* of a graph G if $<N(v)>$ is complete. A set of vertices D in a graph G is a *dominating set* if each vertex of G is dominated by some vertex of D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. For references on domination parameters in graphs see[3,4]. If $e = \{u,v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then we call e a pendent edge, u a leaf and v a support vertex. Let $L(G)$ be the set of all leaves of a graph G. For any connected graph G, a vertex $v \in V(G)$ is called a *cut vertex* of G if $V - v$ is no longer connected. A maximum connected induced subgraph without a cut vertex is called a *block* of G. A graph G is a block graph is every block in G is complete. A set of vertices M in G is called a *geodetic dominating* set if M is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of G is its *geodetic domination number* and is denoted by $\gamma_g(G)$. A geodetic dominating set of size $\gamma_g(G)$ is said to be a γ_g-*set*. The geodetic domination number of a graph was introduced in[9]. A set S of vertices of a connected graph G is an *open geodetic* set if for each vertex v in G either v is an extreme vertex of G and $v \in S$ or v is an internal vertex of a $x-y$ geodesic for some $x,y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number, $og(G)$. A set $S \subset V(G)$ is called an *open geodetic dominating set* of a connected graph G if S is both open geodetic set and dominating set of G. The minimum cardinality of an open geodetic dominating set of G is called *open geodetic domination number* of G and is denoted by $\gamma_{og}(G)$. An open geodetic dominating set of minimum cardinality is called γ_{og}-*set* of G. The following theorems are used in sequel.

Theorem 1.1.[13] Every extreme vertex of a connected graph G belongs to open geodetic dominating set of G. In particular, if the set S of all extreme vertices of G is an open geodetic dominating set of G, then S is the unique minimum open geodetic dominating set of G.

2. THE FORCING OPEN GEODETIC DOMINATION NUMBER OF A GRAPH

Definition 2.1. Let G be a connected graph with at least two vertices and S a minimum open geodetic dominating set of G. A subset T of S is called a *forcing subset* for S if S is the unique minimum open geodetic dominating set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The *forcing open geodetic dominating number* of S, denoted by $f_{\gamma og}(S)$, is the cardinality of a minimum forcing subset of S. The forcing open geodetic dominating number of G denoted by $f_{\gamma og}(G)$ is $f_{\gamma og}(G) = \min\{f_{\gamma og}(S)\}$, where the minimum is taken over all minimum forcing open geodetic dominating sets in G.

Example 2.2. For the graph G in Figure 2.1, $S = \{v_1, v_4\}$ is the unique minimum open geodetic dominating set of G so that $f_{\gamma og}(G) = 0$ and for the graph G given in Figure 2.2, $S_1 = \{v_1, v_4, v_6, v_8\}$ and $S_2 = \{v_1, v_4, v_6, v_7\}$ are the only minimum open geodetic dominating set of G. It is clear that $f_{\gamma og}(S_1) = f_{\gamma og}(S_2) = 1$ so that $f_{\gamma og}(G) = 1$.

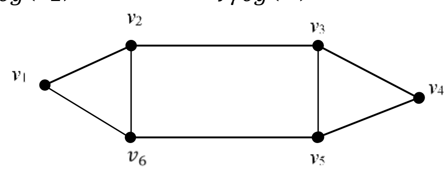

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Figure 2.1

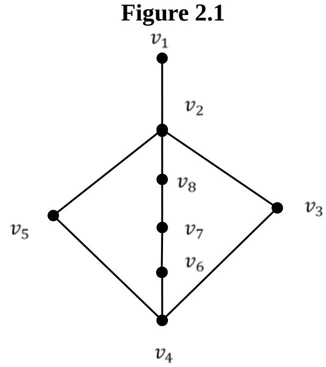

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Figure 2.2

The next theorem follows immediately from the definition of the open geodetic domination number and the forcing open geodetic dominating number of a connected graph G.

Theorem 2.3. For every connected graph G, $0 \leq f_{\gamma og}(G) \leq \gamma_{og}(G) \leq n$. ■

Remark 2.4. The bounds in Theorem 2.3 are sharp. For the graph G given in the Figure 2.1, $f_{\gamma og}(G) = 0$ and for the complete graph K_n ($n \geq 2$), $\gamma_{og}(K_n) = n$. Also all the inequality in the Theorem 2.3 are strict. For the graph G given in Figure 2.2, $n = 8$, $\gamma_{og}(G) = 4$ and $f_{\gamma og}(G) = 1$. Thus $0 < f_{\gamma og}(G) < \gamma_{og}(G) < n$. ■

Theorem 2.5. Let G be a connected graph. Then

i) $f_{\gamma og}(G) = 0$ if and only if G has a unique minimum open geodetic dominating set.

ii) $f_{\gamma og}(G) = 1$ if and only if G has at least two minimum open geodetic dominating set, one of which is a unique minimum open geodetic dominating set containing one of its elements, and

iii) $f_{\gamma og}(G) = \gamma_{og}(G)$ if and only if no minimum open geodetic dominating set of G is the unique minimum open geodetic dominating set containing any of its proper subsets.

Proof. (i) Let $f_{\gamma og}(G) = 0$. Then, by definition, $f_{\gamma og}(S) = 0$ for some minimum open geodetic dominating set S of G so that the empty set ϕ is the minimum forcing subset for S. Hence it follows that S is the unique minimum open geodetic dominating set S of G. The converse is clear.

(ii) Let $f_{\gamma og}(G) = 1$. Then by Theorem 2.5 (i), G has at least two minimum open geodetic dominating sets. Also, since $f_{\gamma og}(G) = 1$, there is a singleton subset M of a minimum open geodetic dominating set S of G such that M is not a subset of any other minimum open geodetic dominating set of G. Thus S is the unique minimum open geodetic dominating sets containing one of its elements. The converse is clear.

(iii) Let $f_{\gamma og}(G) = \gamma_{og}(G)$. Then $f_{\gamma og}(G) = \gamma_{og}(G)$ for every minimum open geodetic dominating set S in G. Also, by Theorem 2.3, $\gamma_{og}(G) \geq 2$ and hence $f_{\gamma og}(G) \geq 2$. Then by Theorem 2.5(i), G has at least two minimum open geodetic dominating sets, and so the empty set ϕ is not a forcing subset for any minimum open geodetic dominating sets of G. Since $f_{\gamma og}(G) = \gamma_{og}(G)$, no proper subset of S is a forcing subset of S. Thus no minimum open geodetic dominating sets of G is the unique minimum open geodetic dominating sets containing any of its proper subsets. Conversely, assume that no minimum open geodetic dominating set of G is the unique minimum open geodetic dominating set containing any of its proper subset of any minimum open geodetic dominating set S other than S is a forcing subset for S. Hence it follows that $f_{\gamma og}(G) = \gamma_{og}(G)$. ■

Definition 2.6. A vertex v of a graph G is said to be a *open geodetic dominating vertex* if v belongs to every minimum open geodetic dominating set of G. ■

Example 2.7. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_4, v_6, v_8\}$ and $S_2 = \{v_1, v_4, v_6, v_7\}$ are the only two minimum open geodetic dominating set of G. Hence the vertices v_1, v_4 and v_6 are the open geodetic dominating vertices of G. ■

Theorem 2.8. Let G be a connected graph and let ξ be the set of relative complements of the minimum forcing subsets in their respective minimum open geodetic dominating set of G. Then $\bigcap_{F\in\xi} F$ is the set of open geodetic dominating vertices of G.

Proof. Let W denote the set of open geodetic dominating vertices of G. We show that $W = \bigcap_{F\in\xi} F$. Let $v \in W$. Then v belongs to every minimum open geodetic dominating set of G. Let $T \subseteq S$ be any minimum forcing subset for any minimum open geodetic dominating set S of G. We claim that $v \notin T$. If $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique minimum x open geodetic dominating set containing T' so that T' is a forcing subset for S with $|T'| < |T|$, which is a contradiction to T, a minimum forcing subset for S. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \bigcap_{F\in\xi} F$ so that $W \subseteq \bigcap_{F\in\xi} F$.

Conversely, let $v \in \bigcap_{F\in\xi} F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is a minimum forcing subset for S. Since F is the relative complement of T in S, we have $F \subseteq S$ and thus $v \in S$ for every S, which implies that v is a open geodetic dominating set vertex of G. Thus $v \in W$ and so $\bigcap_{F\in\xi} F \subseteq W$. Hence $W \subseteq \bigcap_{F\in\xi} F$. ■

Theorem 2.9. Let G be a connected graph, and W the set of all open geodetic dominating vertices of G. Then $f_{\gamma og}(G) \leq \gamma_{og}(G) - |W|$.

Proof. Let S any open geodetic dominating vertices of G. Then $\gamma_{og}(G) = |S|$, $W \subseteq S$ and S is the unique minimum open geodetic dominating containing $S - W$. Thus $f_{\gamma og}(G) \leq |S - W| = |S| - |W| = \gamma_{og}(G) - |W|$. ■

Corollary 2.10. If G is a connected graph with k extreme vertices, then $f_{\gamma og}(G) \leq \gamma_{og}(G) - k$.

Proof. Let k number of extreme vertices of G. Since every extreme vertex of G is open geodetic dominating vertex of G. The result follows from Theorems 1.1. ■

Remark 2.11. The bound in Theorem 2.9 is sharp. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_4, v_6, v_8\}$ and $S_2 = \{v_1, v_4, v_6, v_7\}$ are the only two two γ_{og}(G)–sets so that $\gamma_{og}(G) = 4$ and $f_{\gamma og}(G) = 1$. Also $W = \{v_1, v_4, v_6\}$ is the set of all open geodetic dominating vertices of G and so $f_{\gamma og}(G) = \gamma_{og}(G) - |W|$. Also, the inequality in Theorem 2.9 can be strict. For the graph G given in Figure 2.3,
$S_1 = \{v_1, v_2, v_6, v_7\}$, $S_2 = \{v_2, v_3, v_4, v_7\}$ are the only two γ_{og}(G)-sets of G so that $\gamma_{og}(G) = 4$ and $f_{\gamma og}(G) = 1$.

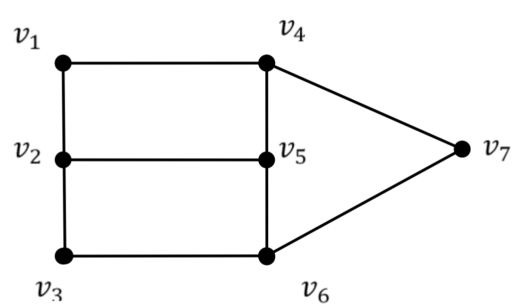

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Figure 2.3

Also, $W = \{v_2, v_7\}$ is the set of all open geodetic dominating vertices of G and so $f_{\gamma og}(G) < \gamma_{og}(G) - |W|$.

In the following we determine the forcing open geodetic dominating number of certain standard graphs.

Theorem 2.12. *For the complete graph* $G = K_n$, $f_{\gamma og}(G) = 0$.

Proof. For $G = K_n$ it follows from Theorem 1.1 that the set of all extreme vertices of G is the unique minimum open geodetic dominating set S of G. Hence it follows from Theorem 2.5(i) that $f_{\gamma og}(\mathrm{G}) = 0$. ∎

Theorem 2.13. *For the non-trivial tree* $f_{\gamma og}(G) = 0$.

Proof. This follows from Corollary 2.10 and Theorem 2.5 (i). ∎

Theorem 2.14. *For a complete bipartite graph* $G = K_{r,s}$ $(2 \le r \le s)$

$$f_{\gamma og}(G) = \begin{cases} 0 & if\ r = s = 2 \\ 2 & if\ 2 = r < s \\ 4 & if\ 3 \le r \le s \end{cases}$$

Proof. Case (i) Let $r = s = 2$. Then $G = C_4$. It is clear that $S = V(G)$ is the unique γ_{og}- set of G.Then by Theorem 2.5(i), $f_{\gamma og}(\mathrm{G}) = 0$.

Case (ii) Let $2 = r < s$. Let U and W be the bipartite sets of G. Let $U = \{u_1, u_2\}$ and $W = \{v_1, v_2, \dots, v_s\}$. Now $S = \{u_1, u_2, x, y\}$ where $x, y \in W$ is γ_{og}-set of G. Since $|S| > 2$, γ_{og}-set is not unique so that $\gamma_{og}(\mathrm{G}) \ge 1$. It is easily verified that no singleton subset of S is a forcing subset of G and so $\gamma_{og}(\mathrm{G}) \ge 2$. Now $S_1 = \{u_1, u_2, v_1, v_2\}$ is a γ_{og}- set of G. Since S_1 is the $\gamma_{og} -$ set containing $\{v_1, v_2\}$, $\{v_1, v_2\}$ is a forcing subset of S_1 and so $f_{\gamma og}(S_1) = 2$. Hence it follows that $f_{\gamma og}(\mathrm{G}) = 2$.
Case (iii) Let $3 \le r \le s$. Let U and W be the bipartite sets of G. Let $U = \{u_1, u_2, \dots, u_r\}$ and $W = \{v_1, v_2, \dots, v_s\}$. Now $S = \{u_i, u_j, v_k, v_t\}$ $i \neq j, k \neq t, 1 \le i \le j \le r$ and $1 \le k, t \le s$ is γ_{og}- set of G. Since $\gamma \ge 3$, γ_{og} -set is not unique so that $f_{\gamma og}(G) \ge 1$. Let S be any γ_{og}- set of G. Then $|S| = 4$. Let $W \subseteq S$. If $|W| = 1$ or $|W| = 2$ or $|W| = 3$, then it is easily verified that W is not a forcing subset of S. Therefore $f_{\gamma og}(S) \ge 4$. This is true for all γ_{og}- set of G. Therefore $f_{\gamma og}(\mathrm{G}) = 4$. ∎

Theorem 2.15. *For every pair* a, b *of integers with* $0 \le a < b$ *and* $b \ge 6$, *there exists a connected graph* G *such that* $f_{\gamma og}(G) = a$ *and* $\gamma_{og}(G) = b$.

Proof. We prove the theorem by considering three cases.

Case 1. $a = 0$.Let $G = K_b$.Then byTheorem 2.5 (i),we have $f_{\gamma og}(\mathrm{G}) = a$ and $\gamma_{og}(G) = b$.
Case 2. $a = 1$.

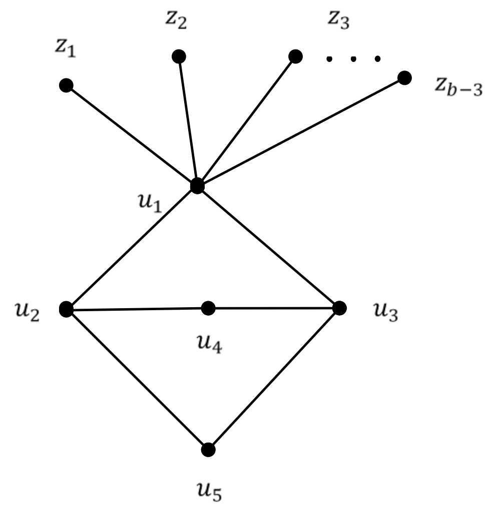

Figure 2.4

Let G be the graph obtained from adding new vertices $u_1, u_2, u_3, u_4, u_5, z_1, z_2, ..., z_{b-3}$ by joining each z_i ($1 \leq i \leq b-3$) with u_1 and u_2 and u_3 with u_1, u_4 and u_5. The graph G is shown in Figure 2.4. Let $S = \{z_1, z_2, \ldots, z_{b-3}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every open geodetic dominating set namely S_1: S $\cup\{u_2, u_3, u_4\}$ and S_2: S $\cup\{u_2, u_4, u_5\}$. Thus γ_{og}(G) = $|S| + 3 =$ b. Since S_1 is the unique open geodetic dominating set containing u_3, it follows from Theorem 2.5(ii) that $f_{\gamma og}$(G) = 1 = a.

Case 3. a ≥ 2. Let F_i: x_i, y_i, z_i be a path of order 3 ($1 \leq i \leq a$). Let P_3: u, v, w be a path of order 3. Let H be the graph obtained from the graphs. P_3 and F_i ($1 \leq i \leq a$) by first adding the za edges ux_i and wz_i for ($1 \leq i \leq a$) and also adding $b-a-2$ new vertices $y_1, y_2, ..., y_{b-a-3}, q$ and joining y_i ($1 \leq i \leq b-a-4$) with w and q with u. Let G be the graph in Figure 2.6 obtained from H by adding new vertex x and joining each x_i and z_i (1 ≤ i ≤ a) with x and also adding the new vertices h and f and joining the edges bx and fv. We prove that $\gamma_{og}(G) = b$ and $f_{\gamma og}$(G) = a. Let $S = \{f, h, q, w_1, w_2, ..., w_{b-a-3}\}$ be the set of all end vertices of G. By Theorem 1.1, every open geodetic dominating set of G contains S. Since the vertices namely $v_1, v_2, ..., v_a$ do not lie on a geodetic path joining any pair of vertices of S, S is not an open geodetic dominating set of G. Let H_i = $\{u_i, w_i\}$ ($1 \leq i \leq a$). We observe that every open geodetic dominating set of G must contains at least one vertex from each H_i (1 ≤ i ≤ a). Also the vertices x_i ($1 \leq i \leq a$) and z_i ($1 \leq i \leq a$) is dominate by x. Then x must lies an every γ_{og}-set of G. Thus $\gamma_{og}(G) \geq b-a+a = b$. Now, Since the set $S_1 = S \cup\{x_1, x_2, ..., , x_a, x\}$ is an open geodetic dominating set of G, it follows that $\gamma_{og}(G) \leq |S| = b$. Thus $\gamma_{og}(G) = b$.

Next we show that $f_{\gamma og}(G) = a$. Since every open geodetic dominating set of G contains $S \cup\{x\}$, it follows from Theorem 2.9, that $f_{\gamma og}$(G) $\leq \gamma_{og}(G)$ - $|S \cup x|$ = $b-(b-$

$a) = a$. Since $\gamma_{og}(G) = b$ and every open geodetic dominating set of G contains S, it is easily seen that every minimum every open geodetic dominating set M is of the form $S \cup \{x\} \cup \{w_1, w_2, \ldots, w_a\}$, Where $w_i \in H_i$ $(1 \leq i \leq a)$. Now, Let T be a proper subset of M with $|T| < a$. Then there is a vertex w_j $(1 \leq j \leq a)$ such that $w_j \notin T$. Let u_j be a vertex of H_j distinct from w_j. Then $Z = (M - \{w_j\}) \cup \{u_j\}$ is a open geodetic dominating set properly containing T. Thus T is not a forcing subset of M. This is true for all minimum open geodetic dominating sets of G and so it follows that $f_{\gamma og}(G) = a$. ■

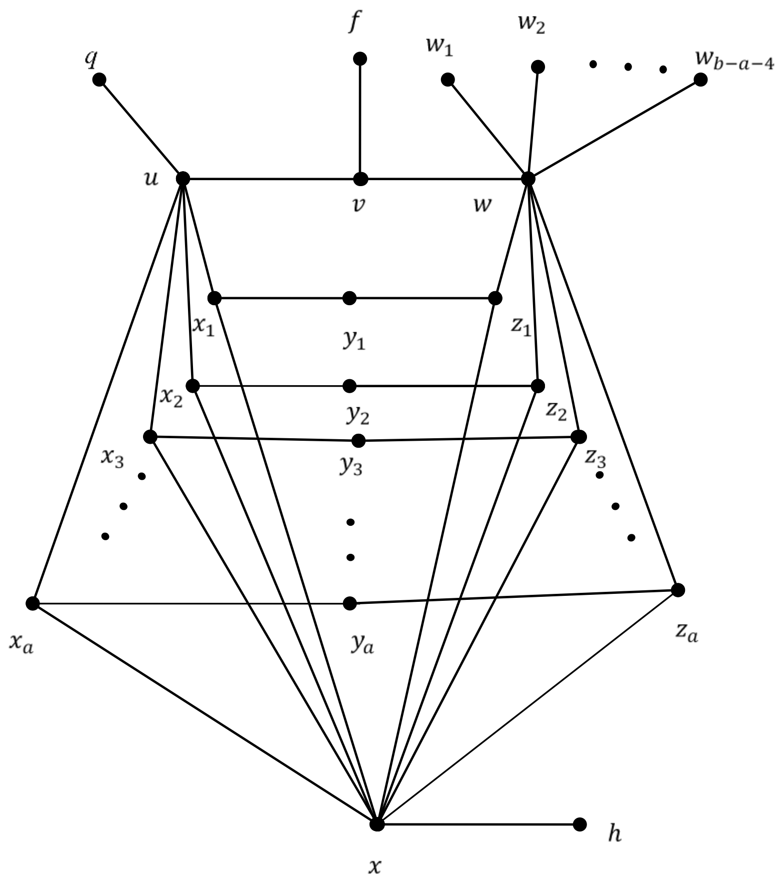

Figure 2.5

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