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Fixed Point Theorem Using Occasionally Weakly Compatible Mappings in Metric Space

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ABSTRACT

The aim of this paper is to present a common fixed point theorem in a metric space which generalizes the result of P.C. Lohani & V.H. Badshah using the weaker conditions such as occasionally weakly compatible mappings and associated sequence in place of compatibility and completeness of the metric space. Moreover the condition of continuity of any one of the mappings is being dropped.

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1. INTRODUCTION

Gerald Jungck gave a common fixed point theorem for commuting mappings, which generalizes the Banach's fixed point theorem. This result was further generalized and extended in various ways by many authors. S. Sessa[5] defined weak commutativity and proved common fixed point theorems for weakly commuting maps. Further G. Jungck[1] introduced the concept of compatible maps which is weaker than weakly commuting maps. Afterwards Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible maps. The concept of occasionally weakly compatible mappings in metric space is introduced by A1-Thagafi and Shahzad[10] which is most general among all the commutativity concepts.

The purpose of this paper is to prove a common fixed point theorem for four self maps in which two pairs are occasionally weakly compatible.

2. DEFINITIONS AND PRELIMINARIES

Definition2.1. Let S and T be two self mappings of a metric space (X,d) then S and T are said to be *commuting* on X if $STx = TSx$ for all x in X .

Definition2.2. Two self maps S and T of a metric space (X,d) are said to be *compatible mappings* if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, S and T are compatible mappings on X, then $d(STx,TSx) = 0$ when $d(Sx,Tx) = 0$, for some x in X.

Definition2.3. Two self maps S and T of a metric space (X,d) are said to be *weakly compatible* if they commute at their coincidence point. i.e., if $Su = Tu$ for some $u \in X$ then $STu = TSu$.

It is clear that every compatible pair is weakly compatible but its converse need not be true.

Definition2.4. Two self maps S and T of a metric space (X,d) are said to be *occasionally weakly compatible* if S and T are commuting at some coincident points. That means S and T are not commuting at all coincidence points.

Weakly compatible mappings are occasionally weakly compatible mappings but converse is not true.

P.C.Lohani and V.H. Badshah[6] proved the following theorem.

Theorem2.5: Let P, Q, S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions

$S(X) \subset Q(X)$ and $T(X) \subset P(X)$ (2.5.1)

$$d(Sx,Ty) \le \alpha \frac{d(Qy,Ty)[1 + d(Px,Sx)]}{[1 + d(Px,Qy)]} + \beta d(Px,Qy) \qquad (2.5.2)$$

for all x, y in X where $\alpha, \beta \ge 0, \alpha + \beta < 1$

one of P, Q, S and T is continuous (2.5.3)

the pairs (S, P) and (T, Q) are compatible on X (2.5.4)

then P, Q, S and T have a unique common fixed point in X.

Now we generalize Theorem2.5 using occasionally weakly compatible mappings and associated sequence.

Associated Sequence2.6[9]: Suppose P, Q, S and T are self maps of a metric space (X,d) satisfying the condition (2.5.1).Then for an arbitrary $x_0 \in X$ such that $Sx_0 = Qx_1$ and for x_1, there exists a point x_2 in X such that $Tx_1 = Px_2$ and so on. Proceeding in the similar manner, we can define a sequence $< y_n >$ in X such that $y_{2n} = Sx_{2n} = Qx_{2n+1}$ and $y_{2n+1} = Px_{2n+2} = Tx_{2n+1}$ for $n \geq 0$ .We shall call this sequence as an "Associated sequence of x_0" relative to the four self maps P,Q,S and T.

Now we prove a lemma which plays an important role in our main Theorem.

Lemma2.7: Let P, Q, S and T be self mappings from a complete metric space (X, d) into itself satisfying the conditions (2.5.1) and (2.5.2).Then the associated sequence $\{y_n\}$ relative to four self maps is a Cauchy sequence in X.

Proof: From the definition of associated sequence (2.6), we have

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \alpha \frac{d(Qx_{2n+1}, Tx_{2n+1})[1 + d(Px_{2n}, Sx_{2n})]}{[1 + d(Px_{2n}, Qy_{2n+1})]} + \beta d(Px_{2n}, Qy_{2n+1})$$

$$= \alpha \frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n-1}, y_{2n})]}{[1 + d(y_{2n-1}, y_{2n})]} + \beta d(y_{2n-1}, y_{2n})$$

$$= \alpha d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})$$

$$(1-\alpha) d(y_{2n}, y_{2n+1}) \leq \beta d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{\beta}{(1-\alpha)} d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq h\, d(y_{2n-1}, y_{2n}) \quad where\ h = \frac{\beta}{1-\alpha}$$

Now

$$d(y_n, y_{n+1}) \leq h d(y_{n-1}, y_n) \leq h^2 d(y_{n-2}, y_{n-1}) \leq .... \leq h^n d(y_0, y_1)$$

for every integer p>0, we get

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+p-1}, y_{n+p})$$

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+p-1}, y_{n+p})$$

$$\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + ... + h^{n+p-1} d(y_0, y_1)$$

$$\leq (h^n + h^{n+1} + ... + h^{n+p-1}) d(y_0, y_1)$$

$$\leq h^n (1 + h + h^2 + ... + h^{p-1}) d(y_0, y_1)$$

since $h<1, h^n \to 0$ as $n \to \infty$ so that $d(y_n, y_{n+p}) \to 0$ .This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space, it converges to a limit, say $z \in X$.
The converse of the Lemma is not true, that is P,Q,S and T are self maps of a metric space (X,d) satisfying (2.5.1) and (2.5.2),even if for any $x_0 \in X$ and for any associated sequence of x_0 converges, the metric space (X,d) need not be complete.

Example2.8: Let $X = \left[\frac{1}{4}, 5\right)$ and with $d(x,y) = |x-y|$ .Define self maps P,Q,S and T of X by

$$Sx = Tx = \begin{cases} x^2 & if\ \frac{1}{4} \le x < 1 \\ 2x-1 & if\ 1 \le x < 5 \end{cases} \quad \text{and} \quad Px = Qx = \begin{cases} \frac{x}{4} & if\ \frac{1}{4} \le x < 1 \\ x^2 & if\ 1 \le x < 5 \end{cases}$$

Then $S(X) = T(X) = \left[\frac{1}{16}, 1\right] \cup\ 1,9$ while $P(X) = Q(X) = \left[\frac{1}{16}, 1\right] \cup [1,25)$

so that $S(X) \subset Q(X)$ and $T(X) \subset P(X)$ proving the condition (2.5.1). It is also easy to prove that the associated sequence $Sx_0, Tx_1, Sx_2, Tx_3, ...Sx_{2n}, Tx_{2n+1}, ...$ converges to 1.

Lemma2.9: Let X be a set, S and P are occasionally weakly compatible self maps of X. If S and P have a unique point of coincidence $w = Sx = Px$,then w is the unique common fixed point of S and P.

Proof: Since S and P are occasionally weakly compatible maps then there exists point $x \in X$ such that $w = Px = Sx$ and $PSx = SPx$ .Thus $PPx = PSx = SPx$ says that Px is also a point of coincidence of S and P. Since the point of coincidence $w = Px$ is unique by hypothesis, $SPx = PPx = Px$ and $w = Px$ is a common fixed point of P and S. Moreover if z is any common fixed point of P and S, then $z = Px = Sx = w$ by the uniqueness of the point of coincidence.

Now we generalize the above Theorem2.5 in the following form.

3. MAIN RESULT

Theorem3.1: Let P,Q,S and T are self maps of a metric space (X,d) satisfying the conditions

$$S(X) \subset Q(X) \text{ and } T(X) \subset P(X) \tag{3.1.1}$$

$$d(Sx,Ty) \le \alpha \frac{d(Qy,Ty)[1+d(Px,Sx)]}{[1+d(Px,Qy)]} + \beta d(Px,Qy) \tag{3.1.2}$$

for all x, y in X where $\alpha, \beta \ge 0, \alpha + \beta < 1$

the pairs (S,P) and (Q,T) both are occasionally weakly compatible . (3.1.3)

Further

the associated sequence relative to four self maps P, Q, S and T such that the sequence $Sx_0, Tx_1, Sx_2, Tx_3......Sx_{2n}, Tx_{2n+1}...$ converges to $z \in X$ as $n \to \infty$ (3.1.4)

then P,Q,S and T have a unique common fixed point in X.

Proof: Using the condition (3.1.4), we have

$Sx_{2n} \to z, Qx_{2n+1} \to z, Tx_{2n+1} \to z \text{ and } Px_{2n+2} \to z \text{ as } n \to \infty$

Since (S,P)and (Q,T) both are occasionally weakly compatible

So there are points x,y in X such that

$Sx = Px \text{ and } Qy = Ty$

Using (3.1.2) we claim $Sx = Qy$

$$d(Sx,Ty) \le \alpha \frac{d(Qy,Ty)[1+d(Px,Sx)]}{[1+d(Px,Qy)]} + \beta d(Px,Qy)$$

using the conditions $Sx = Px$ and $Qy = Ty$, we get

$$d(Sx,Qy) \le \alpha \frac{d(Qy,Qy)[1+d(Sx,Sx)]}{[1+d(Sx,Qy)]} + \beta d(Sx,Qy)$$

$$d(Sx,Qy) \le \beta d(Sx,Qy)$$

$\dfrac{d(Sx,Qy)}{d(Sx,Qy)} \le \beta$, since $\beta \ge 0$,which is a contradiction.

Hence $Sx = Qy$.

Therefore $Sx = Qy = Px = Ty$.

Suppose there is another point of coincidence say, w in X such that $Sz = Pz = w$ then $Sz = Pz = Qy = Ty$, which gives $S_z = Sx$ implies $z = x$.

Hence $w = Sx = Px$ for $w \in X$ is the unique point of coincidence of P and S. By lemma (2.9), w is common fixed point of S and P.Hence $Sw = Pw = w$.

Similarly there exists a common fixed point of Q and T say $v \in X$ such that $v = Qv = Tv$.

Suppose $w \ne v$

put $x = w$ and $y = v$ in the condition (3.1.2), we get

$$d(Sw,Tv) \le \alpha \frac{d(Qv,Tv)[1+d(Pw,Sw)]}{[1+d(Pw,Qv)]} + \beta d(Pw,Qv)$$

$$d(w,v) \le \alpha \frac{d(v,v)[1+d(w,w)]}{[1+d(w,v)]} + \beta d(w,v)$$

$$d(w,v) \le \beta\, d(w,v)$$

$(1-\beta)\, d(w,v) \le 0$ since $\beta \ge 0 \text{ and } \alpha + \beta < 1$

$d(w,v) = 0$ implies $w = v$.

This is a contradiction. Therefore $w = v$ .Hence w is a common fixed point of P, Q, S and T.

Remark3.2: From the example (1.1.12), clearly $x = 1$ and $x = \frac{1}{4}$ are two coincidence points.

If $x = 1$ then $S(1) = P(1) = 1$ and $T(1) = Q(1) = 1$ which gives $SP(1) = 1 = PS(1)$ and $TQ(1) = 1 = QT(1)$ but $SP\left(\frac{1}{4}\right) \neq PS\left(\frac{1}{4}\right)$ and $TQ\left(\frac{1}{4}\right) \neq QT\left(\frac{1}{4}\right)$. Therefore the pairs (S, P) and (T, Q) are occasionally weakly compatible but not weakly compatible. The rational inequality holds for the values of $\alpha, \beta \geq 0, \alpha + \beta < 1$. Moreover '1' is the unique common fixed point of P, Q, S and T.

CONCLUSION

Theorem3.1 is a generalization of Theorem2.5 by virtue of the weaker conditions such as occasionally weakly compatibility of the pairs (S,P) and (Q,T) in place of compatibility of the pairs (S,P) and (Q,T). The continuity of any one of the mappings is being dropped and the convergence of associated sequence relative to four self maps S, P, Q and T is used in place the complete metric space.

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