Journal of Computer and Mathematical Sciences, *Vol.8(10), 533-547 October 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

A Review Article: The Continuous Ridgelet Transform and its Application

V. R. Lakshmi Gorty and Nitu Gupta

Department of Basic Science & Humanities,
SVKM's NMIMS, MPSTME, Vile Parle (W),
Mumbai-400056, INDIA.
email:vr.lakshmigorty@nmims.edu

(Received on: September 27, Accepted: October 23, 2017)

ABSTRACT

This is a relook into the work developed by Candés exposing explicitly the integral representation in $L^2(\mathbb{R}^2)$, arising from translation and modulation of single function ridgelet shown to represent any function in $L^2(\mathbb{R}^2)$ in terms of coherent state. The elementary properties of ridgelet transforms are studied. Fourier transform of ridgelet and ridgelet transforms is also represented. Parseval's relation for the Ridgelet transform and reconstruction formula are also studied in this paper. Relation between ridgelet and radon transforms is relooked with bivariate function. In the last section, the continuous ridgelet transforms are applied to the neural network problem.

2000 Mathematics Subject Classification: 44A05, 42A38, 60E10, 65R10, 82C32.

Keywords: continuous Ridgelet transforms, Radon transforms, Parseval's formula, reconstruction, bivariate, translation operator.

1. INTRODUCTION

In order to clarify the approximation ability, the author in[1] explores a new theorem on an integral transform of ridge functions. In[2], the author in chapter 2 present results regarding the existence and properties of the continuous representation of reproduction formula. The author, in[3] uses the idea of superposition of ridge function to certain kind of multivariate functions. In[4], author has focus on the study of objects defined in two-dimensional space since, on one hand, this case already exhibits the main concepts underlying the ridgelet analysis and, on the other hand, it is a very practical setting because of the connection with image analysis. Donoho explores the relationship between orthonormal ridgelet and true ridge functions $r(x_1\cos\theta + x_2\sin\theta)$ and derived a formula for the ridgelet co-efficient of ridge function in terms of 1-D wavelet coefficients of the ridge profile $r(t)$ in[5]. The comparison of the learning speeds between the two-input-one-output FSNN having 30 harmonic neurons, which was trained with a gradient descent algorithm, and a 2-10-1 sigmoidal network, which was trained with the Levenberg-Marquardt method, is depicted in[12]. In the case of an FSNN, the values of the weights and output signals from neurons take physical interpretations, which is of importance when the networks are used in data stream processing.[12] carried out necessary modifications where each input is associated with a different number of neurons. In this study the relation of ridgelet transform with radon transforms are proved. The application to the study is developed from the field of engineering i.e. neural networks with continuous data streaming.

2. PRELIMINARY RESULTS

Definition 2.1. The Ridgelet transform in two dimensions allows the representation of arbitrary bivariate functions $f(x_1, x_2)$ by superposition of elements of the form[2&13].

$$\psi_{a,b,\theta}(x) = a^{-1/2}\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right) \tag{2.1}$$

$$\psi_{a,b,\theta}(x) = a^{-1/2}\,\psi\left(\frac{x_1(t)\cos\theta + x_2(t)\sin\theta - b}{a}\right) \tag{2.2}$$

$$\vec{u}\cdot\vec{x} = |\vec{u}||\vec{x}|\cos\alpha \tag{2.3}$$

where α is angle between $\vec{u}$ and $\vec{x}$. Assume $|\vec{u}||\vec{x}|\cos\alpha = p.$

Here $a > 0$ is the scale parameter, θ is the orientation parameter and b is the location parameter. x is the vector expressed in scale parameter t which takes different values. These ridgelet are constant along the lines $x_1(t)\cos\theta + x_2(t)\sin\theta$. Using common for θ, b and different scales for a, it is possible to efficiently approximate the singularities along a line. The measure on neuron parameter space is defined by

$$\mu(d\gamma) = \frac{da\,db\,d\theta}{a^3} \tag{2.4}$$

Given an integrable bivariate function $f(x)$, its ridgelet coefficient is define by[11]

$$\Re_{\psi_{a,b,\theta}}(f)(x) = \int_{\mathbb{R}^2} f(x)\overline{\psi_{a,b,\theta}(x)}\,dx \tag{2.5}$$

where $\overline{\psi_{a,b,\theta}(x)}$ is complex conjugate of $\psi_{a,b,\theta}(x)$.

Theorem 2.1: - Let ψ and ϕ are Ridgelets and $f, g \in L^2(\mathbb{R}^2)$, then[8]

i) $\Re_{\psi_{a,b,\theta}}(\alpha f+\beta g)=\alpha\ \Re_{\psi_{a,b,\theta}}(f)+\beta\ \Re_{\psi_{a,b,\theta}}(g)$ for any $a,b\in\mathbb{C}$. (2.6)

Proof: - By the definition of Ridgelet transforms in (2.5),

$$\Re_{\psi_{a,b,\theta}}(\alpha f+\beta g)=\int_{-\infty}^{\infty}(\alpha f+\beta g)(x)\ \overline{\psi_{a,b,\theta}(x)}dx$$

$$=\alpha\int_{-\infty}^{\infty} f(x)\ \overline{\psi_{a,b,\theta}(x)}dx+\beta\int_{-\infty}^{\infty} g(x)\ \overline{\psi_{a,b,\theta}(x)}dx$$

$$=\alpha\ \Re_{\psi_{a,b,\theta}}+\beta\ \Re_{\psi_{a,b,\theta}}$$

$$\Re_{\psi_{a,b,\theta}}(\alpha f+\beta g)=\alpha\ \Re_{\psi_{a,b,\theta}}(f)+\beta\ \Re_{\psi_{a,b,\theta}}(g).$$

ii) $\Re_{\psi_{a,b,\theta}}(T_c f)=\Re_{\psi_{a,(b-c),\theta}}(f)$ (2.7)

where T_c is the translation operator defined by $T_c f(x)=f(x-c)$.

Proof: Since $\psi_{a,b,\theta}(x)=\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)$. Let $\vec{u}\cdot\vec{x}=X(\theta)$ (as $\vec{u}\cdot\vec{x}$ is scalar quantity)

$$\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)=\psi\left(\frac{X(\theta)-b}{a}\right)=\psi_{a,b,\theta}(X) \tag{2.8}$$

From (2.8) and definition of Ridgelet transform (2.5), it follows

$$\Re_{\psi_{a,b,\theta}}(T_c f)=\int_{-\infty}^{\infty}(T_c f)(X)\ \overline{\psi_{a,b,\theta}(X)}dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty}(T_c f)(X)\ \overline{\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)}dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty} f(X-c)\ \overline{\psi\left(\frac{X-b}{a}\right)}dX$$

substituting

$X-c=t;\ dX=dt.$ (2.9)

From (2.9), it follows

$$\Re_{\psi_{a,b,\theta}}(T_c f)=|a|^{-1/2}\int_{-\infty}^{\infty} f(t)\ \overline{\psi\left(\frac{t-(b-c)}{a}\right)}\,dt$$

$$\Re_{\psi_{a,b,\theta}}\left(T_c f\right)=\int_{-\infty}^{\infty} f\left(t\right)\overline{\psi}_{a,(b-c),\theta}\,dt$$

$$=\Re_{\psi_{a,(b-c),\theta}}\left(f\right)$$

$$\Re_{\psi_{a,b,\theta}}\left(T_c f\right)=\Re_{\psi_{a,(b-c),\theta}}\left(f\right)$$

iii) $$\Re_{\psi_{a,b,\theta}}\left(D_c f\right)=\frac{1}{\sqrt{c}}\Re_{\psi_{\frac{a}{c},\frac{b}{c},\theta}}\left(f\right) \tag{2.10}$$

where c is a positive number and D_c is the dilation operator defined by $D_c f\left(t\right)=\frac{1}{c}f\left(\frac{t}{c}\right)$.

Proof: From (2.8) and def. of Ridgelet transform (2.5), it follows

$$\Re_{\psi_{a,b,\theta}}\left(D_c f\right)=\int_{-\infty}^{\infty}\left(D_c f\right)\left(X\right)\overline{\psi_{a,b,\theta}\left(X\right)}\,dX$$

$$=\left|a\right|^{-1/2}\int_{-\infty}^{\infty}\left(D_c f\right)\left(X\right)\overline{\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)}\,dX$$

$$=\left|a\right|^{-1/2}\int_{-\infty}^{\infty}\frac{1}{c}f\left(\frac{X}{c}\right)\overline{\psi\left(\frac{X-b}{a}\right)}\,dX.$$

Substituting $$\frac{X}{c}=t;dX=c\,dt\;; \tag{2.11}$$

From (2.11), it follows

$$\Re_{\psi_{a,b,\theta}}\left(T_c f\right)=\left|a\right|^{-1/2}\frac{1}{c}\int_{-\infty}^{\infty}f\left(t\right)\overline{\psi\left(\frac{ct-b}{a}\right)}c\,dt$$

$$\Re_{\psi_{a,b,\theta}}\left(T_c f\right)=\left|a\right|^{-1/2}\int_{-\infty}^{\infty}f\left(t\right)\overline{\psi\left(\frac{t-b/c}{a/c}\right)}dt$$

$$=\left|\frac{1}{c}\right|^{-1/2}\int_{-\infty}^{\infty}f\left(t\right)\left|a/c\right|^{-1/2}\overline{\psi\left(\frac{t-b/c}{a/c}\right)}dt$$

$$=\frac{1}{\sqrt{c}}\int_{-\infty}^{\infty}f\left(t\right)\overline{\psi}_{\frac{a}{c},\frac{b}{c},\theta}\,dt$$

$$=\frac{1}{\sqrt{c}}\Re_{\psi_{\frac{a}{c},\frac{b}{c},\theta}}\left(f\right)$$

$$\Re_{\psi_{a,b,\theta}}\left(D_c f\right)=\frac{1}{\sqrt{c}}\Re_{\psi_{\frac{a}{c},\frac{b}{c},\theta}}\left(f\right)$$

iv) $\Re_{\psi_{a,b,\theta}}\left(\phi\right)=\overline{\Re_{\phi_{\frac{1}{a},\frac{-b}{c},\theta}}\left(\psi\right)},\ \ a\neq 0.$ (2.12)

Proof: From (2.8) and (2.5), it follows

$$\Re_{\psi_{a,b,\theta}}\left(\phi\right)=\int_{-\infty}^{\infty}\phi\left(X\right)\overline{\psi_{a,b,\theta}\left(X\right)}dX$$

$$=\left|a\right|^{-1/2}\int_{-\infty}^{\infty}\phi\left(X\right)\overline{\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)}dX$$

$$=\left|a\right|^{-1/2}\int_{-\infty}^{\infty}\phi\left(X\right)\overline{\psi\left(\frac{X-b}{a}\right)}dX.$$

Substituting $\frac{X-b}{a}=t; X=at+b; dX=adt$ (2.13)

From (2.13), it follows

$$\Re_{\psi_{a,b,\theta}}\left(\phi\right)=\left|a\right|^{-1/2}\int_{-\infty}^{\infty}\phi\left(at+b\right)\overline{\psi\left(t\right)}a\,dt$$

$$\Re_{\psi_{a,b,\theta}}\left(\phi\right)=\overline{\left|\frac{1}{a}\right|^{-1/2}\int_{-\infty}^{\infty}\psi\left(t\right)\overline{\phi\left(\frac{t+b/a}{1/a}\right)}dt}$$

$$=\overline{\Re_{\phi_{\frac{a}{c},\frac{-b}{c},\theta}}\left(\psi\right)}$$

$$\Re_{\psi_{a,b,\theta}}\left(\phi\right)=\overline{\Re_{\phi_{\frac{1}{a},\frac{-b}{c},\theta}}\left(\psi\right)},\ \ a\neq 0$$

v) $\Re_{\alpha\psi_{a,b,\theta}+\beta\psi_{a,b,\theta}}\left(f\right)=\overline{\alpha}\ \Re_{\psi_{a,b,\theta}}\left(f\right)+\overline{\beta}\ \Re_{\psi_{a,b,\theta}}\left(\psi\right)$ (2.14)

for any α and β.

Proof: - By the definition of Ridgelet transform

$$\Re_{\alpha\psi_{a,b,\theta}+\beta\psi_{a,b,\theta}}\left(f\right)=\int_{-\infty}^{\infty}f\left(t\right)\ \overline{\alpha\psi_{a,b,\theta}+\beta\psi_{a,b,\theta}}\ dt$$

$$=\overline{\alpha}\int_{-\infty}^{\infty}f\left(t\right)\ \overline{\psi_{a,b,\theta}\left(t\right)}\ dt+\overline{\beta}\int_{-\infty}^{\infty}f\left(t\right)\ \overline{\psi_{a,b,\theta}\left(t\right)}\ dt$$

$$=\overline{\alpha}\ \Re_{\psi_{a,b,\theta}}\left(f\right)+\overline{\beta}\ \Re_{\psi_{a,b,\theta}}\left(\psi\right)$$

$$\Re_{\alpha\psi_{a,b,\theta}+\beta\psi_{a,b,\theta}}\left(f\right)=\overline{\alpha}\ \Re_{\psi_{a,b,\theta}}\left(f\right)+\overline{\beta}\ \Re_{\psi_{a,b,\theta}}\left(\psi\right).$$

vi) $\Re_{P\psi_{a,b,\theta}}(Pf)=\Re_{\psi_{a,-b,\theta}}(f)$ (2.15)

where P is the parity operator defined by $Pf(t)=f(-t)$.

Proof: It follows:

$$\Re_{P\psi_{a,b,\theta}}(Pf)=\int_{-\infty}^{\infty}Pf(X)\overline{P\left(\psi_{a,b,\theta}(X)\right)}dX$$

$$=\int_{-\infty}^{\infty}f(-X)\overline{\psi_{a,b,\theta}(-X)}dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty}f(-X)\overline{\psi\left(-\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)\right)}dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty}f(-X)\overline{\psi\left(-\left(\frac{X-b}{a}\right)\right)}dX.$$

Substituting $-X=t;\ dX=-dt.$ (2.16)

From (2.16), it follows

$$\Re_{P\psi_{a,b,\theta}}(Pf)=|a|^{-1/2}\int_{\infty}^{-\infty}f(t)\overline{\psi\left(\frac{t+b}{a}\right)}(-dt)$$

$$\Re_{P\psi_{a,b,\theta}}(Pf)=|a|^{-1/2}\int_{-\infty}^{\infty}f(t)\overline{\psi\left(\frac{t-(-b)}{a}\right)}dt$$

$$\because \Re_{P\psi_{a,b,\theta}}(Pf)=\Re_{\psi_{a,-b,\theta}}(f)$$

vii) $\Re_{T_c\psi_{a,b,\theta}}(f)=\Re_{\psi_{a,\,(b+ca),\theta}}(f)$ (2.17)

Proof: From (2.8) and Ridgelet transform (2.5), it follows

$$\Re_{T_c\psi_{a,b,\theta}}(f)=\int_{-\infty}^{\infty}f(X)\overline{T_c\left(\psi_{a,b,\theta}(X)\right)}dX$$

$$=\int_{-\infty}^{\infty}f(X)\overline{T_c\psi_{a,b,\theta}(X)}dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty}f(X)\overline{T_c\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)}dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty}f(X)\overline{\psi\left(\left(\frac{X-b}{a}\right)-c\right)}dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty}f(X)\overline{\psi\left(\frac{X-b-ac}{a}\right)}dX$$

$$=\Re_{\psi_{a,\,(b+ac),\theta}}(f)$$

$$\Re_{T_c\psi_{a,b,\theta}}(f) = \Re_{\psi_{a,\,(b+ca),\theta}}(f).$$

viii) $$\Re_{D_c\psi_{a,b,\theta}}(f) = \frac{1}{\sqrt{c}}\Re_{\psi_{ac,b,\theta}}(f);\ c>0. \tag{2.18}$$

Proof: From (2.8) and (2.5), it follows

$$\Re_{D_c\psi_{a,b,\theta}}(f) = \int_{-\infty}^{\infty} f(X)\overline{D_c\left(\psi_{a,b,\theta}(X)\right)}dX$$

$$= \int_{-\infty}^{\infty} f(X)\overline{D_c\psi_{a,b,\theta}(X)}dX$$

$$= |a|^{-1/2}\int_{-\infty}^{\infty} f(X)\overline{D_c\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)}dX$$

$$= |a|^{-1/2}\int_{-\infty}^{\infty} f(X)\frac{1}{c}\overline{\psi\left(\left(\frac{X-b}{ac}\right)\right)}dX$$

$$= |ac|^{-1/2}\frac{1}{\sqrt{c}}\int_{-\infty}^{\infty} f(X)\overline{\psi\left(\frac{X-b-ac}{a}\right)}dX$$

$$= \frac{1}{\sqrt{c}}\Re_{\psi_{ac,\,b,\,\theta}}(f).$$

$$\Re_{D_c\psi_{a,b,\theta}}(f) = \frac{1}{\sqrt{c}}\Re_{\psi_{ac,b,\theta}}(f);\ c>0.$$

Theorem 2.2. Let $f, g \in L^2\,(\mathbb{R}^2)$. Then

$$\left\langle f,\psi_{a,b,\theta}\right\rangle = \frac{1}{2\pi}\left\langle \hat{f},\hat{\psi}_{a,b,\theta}\right\rangle \tag{2.19}$$

where $\hat{f}$ and $\hat{\psi}_{a,b,\theta}$ are the Fourier transform of f and $\psi_{a,b,\theta}$ respectively.

Proof: Considering

$$\left\langle \hat{f},\hat{\psi}_{a,b,\theta}\right\rangle = \int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{\psi}_{a,b,\theta}(\xi)}d\xi$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{-i\xi y}f(y)dy\right]\left[\overline{\int_{-\infty}^{\infty}e^{-i\xi x}\psi_{a,b,\theta}(x)dx}\right]d\xi$$

$$= \int_{-\infty}^{\infty}f(y)dy\int_{-\infty}^{\infty}\overline{\psi_{a,b,\theta}(x)}dx\int_{-\infty}^{\infty}e^{i\xi(x-y)}d\xi.$$

But $\delta(x-y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(x-y)}\,d\xi$ [11].

$$\left\langle \hat{f},\hat{\psi}_{a,b,\theta}\right\rangle=\int_{-\infty}^{\infty} f(y)\,dy\int_{-\infty}^{\infty}\overline{\psi_{a,b,\theta}(x)}\,dx\times 2\pi\delta(x-y)$$

$$=2\pi\int_{-\infty}^{\infty}\overline{\psi_{a,b,\theta}(x)}\,dx\int_{-\infty}^{\infty} f(y)\delta(x-y)\,dy$$

$$=2\pi\int_{-\infty}^{\infty}\overline{\psi_{a,b,\theta}(x)}\,f(x)\,dx.$$

where $\int_{-\infty}^{\infty} f(y)\delta(x-y)\,dy=f(x)$. [11].

$$\left\langle \hat{f},\hat{\psi}_{a,b,\theta}\right\rangle=2\pi\int_{-\infty}^{\infty} f(x)\overline{\psi_{a,b,\theta}(x)}\,dx=2\pi\left\langle f,\psi_{a,b,\theta}\right\rangle.$$

Thus $\left\langle f,\psi_{a,b,\theta}\right\rangle=\frac{1}{2\pi}\left\langle \hat{f},\hat{\psi}_{a,b,\theta}\right\rangle$.

Theorem 2.3. (Fourier transform of $\psi_{a,b,\theta}(x)$)

For any $f\in L^2(\mathbb{R}^2)$,

$$\hat{\psi}_{a,b,\theta}(\omega)=F\left[\psi_{a,b,\theta}(x)\right](\omega)=|a|^{-1/2}e^{-i\omega b}\hat{\psi}(a\omega) \tag{2.20}$$

where $\hat{\psi}_{a,b,\theta}$ denotes the Fourier transform of $\psi_{a,b,\theta}$ and $F\left[\psi_{a,b,\theta}\right]$ as well.

Proof: Considering from the (2.20) gives:

$$\hat{\psi}_{a,b,\theta}(\omega)=F\left[\psi_{a,b,\theta}(x)\right](\omega)$$

$$=\int_{-\infty}^{\infty} e^{-i\omega X}\psi_{a,b,\theta}(x)\,dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty} e^{-i\omega X}\psi\left(\frac{\vec{u}\cdot\vec{x}-b}{a}\right)dX$$

$$=|a|^{-1/2}\int_{-\infty}^{\infty} e^{-i\omega X}\psi\left(\frac{X(\theta)-b}{a}\right)dX$$

Let $\frac{X-b}{a}=t;\ dX=a\,dt$. (2.21)

Substituting from (2.21) it is obtained as:

$$\hat{\psi}_{a,b,\theta}(\omega) = F\left[\psi_{a,b,\theta}(x)\right](\omega)$$

$$= |a|^{-1/2} \int_{-\infty}^{\infty} e^{-i\omega(at+b)} \psi(t)\, a\, dt$$

$$= |a|^{1/2} e^{-i\omega b} \int_{-\infty}^{\infty} e^{-i\omega at} \psi(t)\, dX$$

$$= |a|^{-1/2} e^{-i\omega b} \hat{\psi}(a\omega).$$

Theorem 2.4. (Fourier transform of Ridgelet transform)

For any $f \in L^2(\mathbb{R}^2)$,

$$F\left[R_{\psi_{a,b,\theta}} f\right] = |a|^{1/2}\, \hat{f}(\omega)\overline{\hat{\psi}(a\omega)}. \tag{2.22}$$

Proof: Using (2.19), it follows:

$$\Re_{\psi_{a,b,\theta}} f = \left\langle f, \psi_{a,b,\theta} \right\rangle$$

$$= \frac{1}{2\pi}\left\langle \hat{f}, \hat{\psi}_{a,b,\theta} \right\rangle$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{\psi}_{a,b,\theta}(\omega)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\overline{|a|^{1/2} e^{-i\omega b}\hat{\psi}(a\omega)}\, d\omega$$

$$\Re_{\psi_{a,b,\theta}} f = \frac{|a|^{1/2}}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega b}\, \overline{\hat{\psi}(a\omega)}\, d\omega. \tag{2.23}$$

$$\Re_{\psi_{a,b,\theta}} f = |a|^{1/2} F^{-1}\left\{\hat{f}(\omega)\overline{\hat{\psi}(a\omega)}\right\}$$

$$F\left[R_{\psi_{a,b,\theta}} f\right] = |a|^{1/2}\, \hat{f}(\omega)\overline{\hat{\psi}(a\omega)}.$$

(2.22) follows the Parseval's relation for the Ridgelet transform as:

Theorem 2. 5. (Parseval's relation for the Ridgelet transform)

Let $\psi \in L^2(\mathbb{R}^2)$which satisfy the admissibility condition; $k_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|}{|\omega|} d\omega < \infty$, then for any $f, g \in L^2(\mathbb{R}^2)$ is given by

$$\langle f, g \rangle = c_\psi \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \Re_{\psi_{a,b,\theta}}(f) \overline{\Re_{\psi_{a,b,\theta}}(g)} \frac{da}{a^3} \, db \, d\theta \tag{2.24}$$

where $c_\psi = \pi(2\pi)^{-2} k_\psi^{-1}$

Proof: From (2.23) it is given as:

$$\Re_{\psi_{a,b,\theta}} f = \frac{|a|^{1/2}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega b} \overline{\hat{\psi}(a\omega)} \, d\omega. \tag{2.25}$$

And

$$\overline{\Re_{\psi_{a,b,\theta}} g} = \frac{|a|^{1/2}}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{g}(\rho)} \, e^{-i\rho b} \, \hat{\psi}(a\rho) \, d\rho \cdot \tag{2.26}$$

Substituting (2.25) and (2.26) in right hand-side of (2.24) gives

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \Re_{\psi_{a,b,\theta}}(f) \overline{\Re_{\psi_{a,b,\theta}}(g)} \frac{da}{a^3} \, db \, d\theta$$

$$= \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \left[\frac{|a|^{1/2}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega b} \, \overline{\hat{\psi}(a\omega)} \, d\omega \right] \left[\frac{|a|^{1/2}}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{g}(\rho)} e^{-i\rho b} \, \hat{\psi}(a\rho) \, d\rho \right] \frac{da}{a^3} \, db \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{da}{|a|^2} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega) \overline{\hat{g}(\rho)} \, \overline{\hat{\psi}(a\omega)} \hat{\psi}(a\rho) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ib(\omega-\rho)} db \, d\omega \, d\rho \right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{da}{|a|^2} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega) \overline{\hat{g}(\rho)} \, \overline{\hat{\psi}(a\omega)} \hat{\psi}(a\rho) \delta(\rho - \omega) \, d\omega \, d\rho \right).$$

As $\delta(\rho - \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ib(\omega-\rho)} db,$

Let $\rho = \omega$ and $\omega > 0$

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \Re_{\psi_{a,b,\theta}}(f) \overline{\Re_{\psi_{a,b,\theta}}(g)} \frac{da}{a^3} \, db \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{da}{|a|^2} \left(2 \int_0^{\infty} \hat{f}(\omega) \overline{\hat{g}(\omega)} \left| \hat{\psi}(a\omega) \right|^2 d\omega \right)_{\omega > 0}$$

$$= \frac{1}{\pi} (2\pi) \int_0^{\infty} \hat{f}(\omega) \overline{\hat{g}(\omega)} \left[\int_0^{\infty} \frac{\left| \hat{\psi}(a\omega) \right|^2}{|a|^2} da \right] d\omega$$

$$= 2 \int_0^{\infty} \hat{f}(\omega) \overline{\hat{g}(\omega)} k_\psi \, d\omega$$

where $k_{\psi}=\int_{0}^{\infty}\frac{|\hat{\psi}(a\omega)|^{2}}{|a|^{2}}da.$ (2.27)

$$\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\overline{\Re_{\psi_{a,b,\theta}}(g)}\frac{da}{a^{3}}\,db\,d\theta=\frac{1}{\pi}2\pi k_{\psi}\left\langle \hat{f},\hat{g}\right\rangle$$

$$=\frac{1}{\pi}2\pi k_{\psi}2\pi\left\langle f,g\right\rangle$$

$$=\frac{1}{\pi}(2\pi)^{2}k_{\psi}\left\langle f,g\right\rangle$$

$$=\frac{1}{c_{\psi}}\left\langle f,g\right\rangle.$$

Hence,

$$\left\langle f,g\right\rangle=c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\overline{\Re_{\psi_{a,b,\theta}}(g)}\frac{da}{a^{3}}\,db\,d\theta.$$

In particular $g=\overline{f}$ as in[9], it is given as

$$\|f\|^{2}=c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\overline{\Re_{\psi_{a,b,\theta}}(f)}\frac{da}{a^{3}}\,db\,d\theta$$

$$\|f\|^{2}=c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\left|\Re_{\psi_{a,b,\theta}}(f)\right|^{2}\frac{da}{a^{3}}\,db\,d\theta. \quad (2.28)$$

Theorem 2. 6. (Reconstruction Formula)

If $f\in L^{2}$ ($\mathbb{R}^2$), then

$$f(x)=c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\psi_{a,b,\theta}(x)\frac{da}{a^{3}}\,db\,d\theta. \quad (2.29)$$

Proof: For any $f\in L^{2}$ ($\mathbb{R}^2$), by (2.24)

$$\left\langle f,g\right\rangle=c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\overline{\int_{-\infty}^{\infty}\overline{\psi_{a,b,\theta}}g(x)\,dx}\frac{da}{a^{3}}\,db\,d\theta$$

$$=c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\int_{-\infty}^{\infty}\psi_{a,b,\theta}\overline{g(x)}\,dx\frac{da}{a^{3}}\,db\,d\theta$$

$$=\int_{-\infty}^{\infty}c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\psi_{a,b,\theta}\frac{da}{a^{3}}\,db\,d\theta\,\overline{g(x)}\,dx$$

$$\left\langle f,g\right\rangle=\left\langle c_{\psi}\int_{0}^{2\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\Re_{\psi_{a,b,\theta}}(f)\psi_{a,b,\theta}\frac{da}{a^{3}}\,db\,d\theta,g(x)\right\rangle$$

i.e. $f(x) = c_\psi \int_0^{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \Re_{\psi_{a,b,\theta}}(f) \psi_{a,b,\theta}(x) \frac{da}{a^3} db\, d\theta.$

Since g is an arbitrary element of L^2 ($\mathbb{R}^2$), the reconstruction formula is proved.

3. THE RELATION BETWEEN RADON AND RIDGELET TRANSFORM

Property 3.1: If $f \in L^2$ ($\mathbb{R}^2$), then (by reference[6 & 7]

$$\left\langle R\{f(x,y)\}(t,u_1,u_2),\ \psi_{a,b}(t)\right\rangle = \Re_{\psi_{a,b,\theta}} f(x,y) \tag{3.1}$$

where $R\{f(x,y)\}(t,u_1,u_2)$ is Radon transform of $f(x,y)$ and $\psi_{a,b}(t)$ is wavelet.

Proof: Considering

$$\left\langle R\{f(x,y)\}(t,u_1,u_2),\ \psi_{a,b}(t)\right\rangle$$

$$= \int_{-\infty}^{\infty} R\{f(x,y)\}(t,u_1,u_2)\overline{\psi_{a,b}(t)}\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(t-u_1x-u_2y)|a|^{-1/2}\overline{\psi\left(\frac{t-b}{a}\right)}dx\, dy\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left[\int_{-\infty}^{\infty}\delta(t-u_1x-u_2y)|a|^{-1/2}\overline{\psi\left(\frac{t-b}{a}\right)}dt\right]dx\, dy.$$

Let $\frac{t-b}{a} = p;\ dt = a\, dp.$ (3.2)

Substituting from (3.2) is obtained as:

$$\left\langle R\{f(x,y)\}(t,u_1,u_2),\ \psi_{a,b}(t)\right\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left[\int_{-\infty}^{\infty}|a|^{-1/2}\overline{\psi(p)}\delta(ap+b-u_1x-u_2y)a\, dp\right]dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left[\int_{-\infty}^{\infty}|a|^{-1/2}\overline{\psi(p)}\frac{1}{a}\delta\left(p-\left(\frac{-b+u_1x+u_2y}{a}\right)\right)a\, dp\right]dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\left[\int_{-\infty}^{\infty}|a|^{-1/2}\overline{\psi(p)}\delta\left(p-\left(\frac{-b+u_1x+u_2y}{a}\right)\right)dp\right]dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\overline{\psi\left(\frac{u_1x+u_2y-b}{a}\right)}\, dx\, dy.$$

$$\left[\because \int_{-\infty}^{\infty} f(y)\,\delta(x-y)\,dy = f(x)\right]$$

$$\left\langle R\{f(x,y)\}(t,u_1,u_2),\ \psi_{a,b}(t)\right\rangle = \Re_{\psi_{a,b,\theta}} f(x,y). \tag{3.3}$$

4. APPLICATION

The networks trained to map the $f(x,y)=e^{\left(-(x-2)^2-(y-1.5)^2\right)}-e^{\left(-(x-4)^2-(y-4)^2\right)}$ occur in Neural network during data stream processing as in[12], find the Ridgelet transform of such a function. The Ridgelet transform of this function $f(x,y)$ is given by

$$\Re_{\psi_{a,b,\theta}}\left(f(x,y)\right)=|a|^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\overline{\psi_{a,b,\theta}\left(\frac{u_1x+u_2y-b}{a}\right)}\,dx\,dy$$

From definition of Radon transform[11] given by

$$R\{f(x,y)\}(t,u_1,u_2)=\int_{-\infty}^{\infty} f(t,s)\,ds \tag{4.1}$$

$$\begin{aligned} f(x,y)&=e^{\left(-(x-2)^2-(y-1.5)^2\right)}-e^{\left(-(x-4)^2-(y-4)^2\right)}\\ &=e^{\left(-t^2+4t\cos\theta+3t\sin\theta-6.25\right)}\times e^{\left(-s^2-4s\sin\theta+3s\cos\theta\right)}-e^{\left(-t^2+8t\cos\theta+8t\sin\theta-32\right)}\times e^{\left(-s^2-8s\sin\theta+8s\cos\theta\right)}.\end{aligned}$$

where

$$x=t\cos\theta-s\sin\theta \tag{4.2}$$

$$y=t\sin\theta+s\cos\theta \tag{4.3}$$

$$x^2+y^2=t^2+s^2. \tag{4.4}$$

From (4.1) to (4.4) it follows

$$\begin{aligned} R\{f(x,y)\}(t,u_1,u_2)&=e^{\left(-t^2+4t\cos\theta+3t\sin\theta-6.25\right)}\times\int_{-\infty}^{\infty} e^{\left(-s^2-4s\sin\theta+3s\cos\theta\right)}\,ds\\ &\quad-e^{\left(-t^2+8t\cos\theta+8t\sin\theta-32\right)}\times\int_{-\infty}^{\infty} e^{\left(-s^2-8s\sin\theta+8s\cos\theta\right)}\,ds.\end{aligned}$$

$$\begin{aligned} R\{f(x,y)\}(t,u_1,u_2)&=\sqrt{\pi}e^{\left(\frac{1}{4}(3\cos\theta-4\sin\theta)^2\right)}\times e^{\left(-t^2+4t\cos\theta+3t\sin\theta-6.25\right)}\\ &\quad-\sqrt{\pi}e^{(16-16\sin2\theta)}\times e^{\left(-t^2+8t\cos\theta+8t\sin\theta-32\right)}\end{aligned} \tag{4.5}$$

(3.3) is represented as:

$$\Re_{\psi_{a,b,\theta}} f(x,y)=\int_{-\infty}^{\infty} R\{f(x,y)\}(t,u_1,u_2)\,\psi_{a,b}(t)\,dt \tag{4.6}$$

From (4.6), it follows

$$\Re_{\psi_{a,b,\theta}} f(x,y) = \int_{-\infty}^{\infty} \sqrt{\pi} e^{\left(\frac{1}{4}(3\cos\theta - 4\sin\theta)^2\right)} \times e^{\left(-t^2 + 4t\cos\theta + 3t\sin\theta - 6.25\right)} \psi_{a,b}(t)\,dt - \int_{-\infty}^{\infty} \sqrt{\pi} e^{(16-16\sin 2\theta)} \times e^{\left(-t^2+8t\cos\theta+8t\sin\theta-32\right)} \psi_{a,b}(t)\,dt \tag{4.7}$$

Considering $\psi_{a,b}(t) = (1-t^2)e^{-at^2/2}$ Mexican Hat wavelet[9].

At $a=1, \quad \psi_{a,b}(t) = (1-t^2)e^{-t^2/2}.$

(4.7) is obtained as:

$$\Re_{\psi_{a,b,\theta}} f(x,y) = \int_{-\infty}^{\infty} \left(\sqrt{\pi} e^{\left(\frac{1}{4}(3\cos\theta - 4\sin\theta)^2\right)} \times e^{\left(-t^2 + 4t\cos\theta + 3t\sin\theta - 6.25\right)} \right)(1-t^2)e^{-t^2/2}\,dt$$
$$- \int_{-\infty}^{\infty} \left(\sqrt{\pi} e^{(16-16\sin 2\theta)} \times e^{\left(-t^2+8t\cos\theta+8t\sin\theta-32\right)} \right)(1-t^2)e^{-t^2/2}\,dt$$
$$= \sqrt{\pi} e^{\left(\frac{1}{4}(3\cos\theta-4\sin\theta)^2 - 6.25\right)} \int_{-\infty}^{\infty} \left(e^{\left(-t^2+4t\cos\theta+3t\sin\theta\right)} \right)(1-t^2)e^{-t^2/2}\,dt$$
$$- \sqrt{\pi} e^{(16-16\sin 2\theta - 32)} \int_{-\infty}^{\infty} \left(e^{\left(-t^2+8t\cos\theta+8t\sin\theta\right)} \right)(1-t^2)e^{-t^2/2}\,dt$$

$$\Re_{\psi_{a,b,\theta}} f(x,y) = \sqrt{\pi} e^{\left(\frac{1}{4}(3\cos\theta-4\sin\theta)^2 - 6.25\right)} \int_{-\infty}^{\infty} \left(e^{\left(-3t^2/2+4t\cos\theta+3t\sin\theta\right)} \right)(1-t^2)\,dt$$
$$- \sqrt{\pi} e^{(16-16\sin 2\theta - 32)} \int_{-\infty}^{\infty} \left(e^{\left(-3t^2/2+8t\cos\theta+8t\sin\theta\right)} \right)(1-t^2)e^{-t^2/2}\,dt$$
$$= \sqrt{\pi} e^{\left(\frac{1}{4}(3\cos\theta-4\sin\theta)^2 - 6.25\right)} \left[\begin{array}{l} \left(\sqrt{2\pi/3}\, e^{\left(\frac{1}{6}(3\sin\theta+4\cos\theta)^2\right)} \right) \\ - \left(\sqrt{2\pi/3}\, \frac{1}{9} e^{\left(\frac{1}{6}(3\sin\theta+4\cos\theta)^2\right)} \left(3 + (4\cos\theta + 3\sin\theta)^2\right) \right) \end{array} \right]$$
$$- \sqrt{\pi} e^{(16-16\sin 2\theta - 32)} \left[\begin{array}{l} \left(\sqrt{2\pi/3}\, e^{\left(\frac{32}{3}(\sin\theta+\cos\theta)^2\right)} \right) \\ - \left(\sqrt{2\pi/3}\, \frac{1}{9} e^{\left(\frac{32}{3}(\sin\theta+\cos\theta)^2\right)} \left(3 + 64(\cos\theta + \sin\theta)^2\right) \right) \end{array} \right]$$
$$= \sqrt{\pi} e^{\left(\frac{1}{4}(3\cos\theta-4\sin\theta)^2 - 6.25\right)} \left[\left(\sqrt{2\pi/3} e^{\left(\frac{1}{6}(3\sin\theta+4\cos\theta)^2\right)} \right) \left(1 - \frac{1}{9}\left(3 + (4\cos\theta + 3\sin\theta)^2\right) \right) \right]$$

$$-\sqrt{\pi}e^{(16-16\sin 2\theta-32)}\left[\left(\sqrt{2\pi/3}\,e^{\left(\frac{32}{3}(\sin\theta+\cos\theta)^2\right)}\right)\left(1-\frac{1}{9}\left(3+64(\cos\theta+\sin\theta)^2\right)\right)\right]$$

$$\Re_{\psi_{a,b,\theta}} f(x,y)=\sqrt{\pi}e^{\left(\frac{1}{4}(3\cos\theta-4\sin\theta)^2-6.25\right)}\left[\left(\sqrt{2\pi/3}e^{\left(\frac{1}{6}(3\sin\theta+4\cos\theta)^2\right)}\right)\left(1-\frac{1}{9}\left(3+(4\cos\theta+3\sin\theta)^2\right)\right)\right]$$
$$-\sqrt{\pi}e^{(16-16\sin 2\theta-32)}\left[\left(\sqrt{2\pi/3}\,e^{\left(\frac{32}{3}(\sin\theta+\cos\theta)^2\right)}\right)\left(1-\frac{1}{9}\left(3+64(\cos\theta+\sin\theta)^2\right)\right)\right].$$

5. REFERENCES

1. Murata, Noboru, An integral representation of functions using three-layered networks and their approximation bounds, Neural Networks, 947-956 (1996).
2. Candès, E., Ridgelets, Theory and applications, Diss. PhD thesis, Stanford University (1998).
3. Candès, Emmanuel J., Harmonic analysis of neural networks, *Applied and Computational Harmonic Analysis*, 197-218 (1999).
4. Candès, Emmanuel J., and David L. Donoho, Ridgelets: A key to higher-dimensional intermittency? , Philosophical Transactions of the Royal Society of London A: *Mathematical*, *Physical and Engineering Sciences*, 2495-2509 (1999).
5. Donoho, David L, Ridge functions and orthonormal ridgelets, *Journal of Approximation Theory*, 143-179 (2001).
6. Candès, Emmanuel J., and David L. Donoho. Ridgelets: A key to higher-dimensional intermittency?. Philosophical Transactions of the Royal Society of London A: *Mathematical, Physical and Engineering Sciences*, 2495-2509 (1999).
7. Granai, Lorenzo, Fulvio Moschetti, and Pierre Vandergheynst. Ridgelet transform applied to motion compensated images. Multimedia and Expo, ICME'03. Proceedings, vol. 1. *IEEE*, (2003).
8. Debnath, Lokenath, and Firdous Ahmad Shah, Wavelet transforms and their applications, Boston, Birkhäuser, (2002).
9. Pathak, Ram Shankar, The wavelet transform, vol. 4, *Springer Science & Business Media*, (2009).
10. Soman, K. P, Insight into wavelets: From theory to practice, PHI Learning Pvt Ltd, (2010).
11. Debnath, Lokenath, and Dambaru Bhatta, Integral transforms and their applications, CRC press, (2014).
12. Halawa, Krzysztof, Determining the weights of Fourier-series neural network on the basis of the multidimensional discrete Fourier transform, *Applied Mathematics and Computer Science*, 369-375 (2008).
13. Starck, Jean-Luc, Emmanuel J. Candès, and David L. Donoho, The curvelet transform for image de-noising, *IEEE Transactions on image processing*, 670-684 (2002).