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Metric Dimension of Certain Mesh Derived Graphs

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ABSTRACT

Let $G(V,E)$ be a graph and let $W \subset V$. Then W is called a metric basis, if for every pair of vertices u, $v \in V$, there exists a vertex $w \in W$ such that the length of a shortest path from w to u is different from the length of a shortest path from w to v; that is $d(w,u) \neq d(w,v)$. The minimum cardinality of a metric basis is called metric dimension and is denoted by $\beta(G)$; the members of a metric basis are called landmarks. In this paper we introduce some new mesh derived architectures and investigate the metric dimension for those graphs.

Keywords: Metric basis, metric dimension, landmarks, mesh derived architectures**.**

1. INTRODUCTION

Let $G(V,E)$ be a graph and let $W \subset V$. Then W is called a metric basis, if for every pair of vertices u, $v \in V$, there exists a vertex $w \in W$ such that the length of a shortest path from w to u is different from the length of a shortest path from w to v; that is $d(w,u) \neq d(w,v)$. The minimum cardinality of a metric basis is called metric dimension and is denoted by $\beta(G)$; the members of a metric basis are called landmarks[12].

This problem has application in the field of robotics. Robot is a self-governing, programmable electromechanical device used in industry and in scientific research to perform a task or a limited repertory of tasks. Robots are a subcategory of automated devices. Although no generally recognized criteria exist that distinguish them from other automated systems, robots tend to be more versatile and adaptable, or reprogrammable, than the less sophisticated devices. They offer the advantages of being able to perform more quickly, cheaply and accurately than humans in conducting set routines. They are capable of operating in locations or under conditions hazardous to human health, ranging from areas of the factory floor to the ocean depths and outer space.

Navigation of a robot can be studied in a graph-structured framework. The navigating agent can be assumed to be a point robot which moves from node to node of a `graph space'[10]. For this robot there is neither the concept of direction nor that of visibility. But it is assumed that it can send a signal to find out how far it is from each among a set of fixed landmarks. Thus its position on the graph is uniquely determined. This suggests the following problem.

The problem of finding the minimum metric dimension MMD of a graph was first studied by Harary and Melter. They gave a characterization for the metric dimension of trees. The concepts of metric basis and minimum metric basis have appeared in the literature under different names as early as 1975. Slater in[16] and later in[17] had called these sets as locating sets and reference sets respectively. Slater called the cardinality of a reference set as the location number of *G*. He described the usefulness of these ideas when working with sonar and loran stations.

Harary and Melter[8] used the terms metric basis and metric dimension, rather than reference set and location number. The metric dimension problem for grids induced by lattice points in the plane was studied by Melter and Tomescu[13]. A simple but interesting result in this topic is that a graph has minimum metric dimension 1 if and only if it is a path.

In this paper we investigate the minimum metric dimension problem for two architectures derived from the standard $m \times n$ mesh. By making each 4-cycle in an $m \times n$ mesh into a complete subgraph we obtain an architecture called the extended mesh denoted by $EX(m,n)$. See Figure 1(a). We place a vertex in each bounded face of an m×n mesh and join it to the corner vertices of the face. We call this architecture as an enhanced mesh and denote it by $EN(m,n)$. See Figure 1(b).

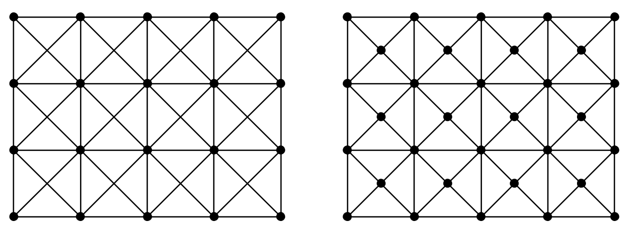

Figure 1: (a) Extended Mesh *EX*(4,5) (b) Enhanced Mesh *EN*(4,5)

2. AN OVERVIEW OF THE PAPER

The problem of finding the minimum metric dimension of a graph was first studied by Harary and Melter[8]. They gave a characterization for the metric dimension of trees. The metric dimension problem for grids induced by lattice points in the plane was studied by Melter and Tomescu[13]. This result was generalized by Samir Khuller *et al.*[12] and they proved that the metric dimension of *d*- dimensional grids is *d*.

A simple but interesting result in this topic is that a graph has metric dimension 1 if and only if it is a path. The problem of computing the metric dimension of trees is solved efficiently in linear time[12]. For a graph *G* with *p* vertices it is clear that $1 \leq \beta(G) \leq p-1$. For a complete graph K_p, a cycle C_p and a complete bipartite graph $K_{m,n}$, the metric dimensions are given by $\beta(K_p) = p-1$, $\beta(C_p) = 2$ and $\beta(K_{m,n}) = m + n - 2$ respectively[8]. The minimum metric dimension problem was investigated[1] for a class of Petersen graph $P(n, 2)$, n even where the metric dimension $\beta(G)$ is given by $\beta(G) = 3$. Garey and Johnson[6] proved that this problem is NP-complete for general graphs by a reduction from 3-dimensional matching.

3. MESH DERIVED ARCHITECTURE

The mesh networks have been recognized as versatile interconnection networks for massively parallel computing. This is mainly due to the fact that these families of networks have topologies which reflect the communication pattern of a wide variety of natural problems. Mesh/torus-like low-dimensional networks have recently received a lot of attention for their better scalability to larger networks, as opposed to more complex networks such as hypercubes.

3.1 Extended Mesh *EX*(*m*,*n*)

We begin with the concept of the neighbourhood of a vertex. For any vertex u and any positive integer r, the r-neighbourhood of u is given by $N_r(u) = \{v: d(u,v) = r\}$.

Let a, b, c and d denote the vertices of $EX(m,n)$ in (1,1), (1,n), (m,1) and (m,n) positions respectively. It is clear that the vertices in $N_r(a)$ induce a path made up of segments in two directions, say the x and y directions. The segment of path along the x direction is denoted by P_x and that along the y direction is denoted by P_y. Thus $N_r(a) = P_x \circ P_y$. Here the notation $\circ$ stands for concatenation of paths. See Figure 2. Similar notions apply to neighbourhoods of vertices b, c and d.

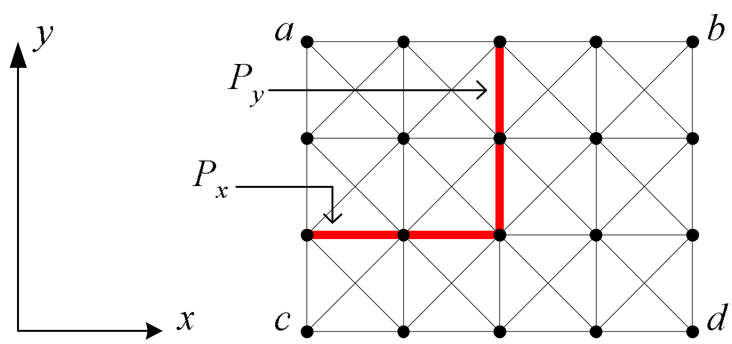

Figure 2: The 2-neighbourhood of the vertex *a* being made up of segments of path along the *x* and *y* directions

We first provide a lower bound for the minimum metric dimension of the extended mesh $EX(m,m)$.

Theorem (due to Samir Khuller[12]): Let $G=(V,E)$ be a graph with minimum metric dimension 2 and let $\{a,b\}\subset V$ be a minimum metric basis in G. Then the following are true:

(1) There is a unique shortest path P between a and b.
(2) The degrees of a and b are at most 3.
(3) Every other node on P has degree at most 5.

The following theorem gives a lower bound on the diameter of a graph with minimum metric dimension k.

Lemma 1: Let G be an extended mesh $EX(m,m)$. Then $\beta(G)>2$.

Proof: By Theorem (Samir Khuller), if G has a metric basis of cardinality 2, then it could possibly be pairs of corner vertices of the extended mesh, since all other vertices have degree more than 3. The vertices c and d are equidistant from both a and b. That is $d(a,c) = d(a,d) = m$ and $d(b,c) = d(b,d) = m$. Hence $\{a, b\}$ is not a metric basis. Now consider the vertices a and d. The vertices b and c are equidistant from a and d. Hence $\{a, d\}$ is not a metric basis. The other possibilities namely $\{a, c\}$, $\{b, c\}$, $\{b, d\}$ and $\{c, d\}$ are ruled out by the symmetrical nature of $EX(m,m)$.

The following result is obvious from the structure of the extended mesh.

Lemma 2: Let G be an extended mesh $EX(m,m)$. Then for any r_1, r_2, $0<r_1,r_2\leq m$

i) $N_{r_1}(a)\cap N_{r_2}(b)$ is either empty, a singleton or induces a path P_x or P_y.

ii) $|N_{r_1}(b)\cap N_{r_2}(c)| \leq 2$; in case $N_{r_1}(b)\cap N_{r_2}(c)=\{u,v\}$, then u and v do not induce a P_x or P_y.

Proof: (1) If $r_1<[(m/2)]$ and $r_2\leq[(m/2)]$ then $N_{r_1}(a)\cap N_{r_2}(b)=\Phi$. The same conclusion holds when r_1 and r_2 are interchanged. If $r_1\leq[(m/2)]$ and $r_2>[(m/2)]$ then $N_{r_1}(a)\cap N_{r_2}(b)$ is a singleton. See Figure 3. If $r_1\geq[(m/2)]$ and $r_2\geq[(m/2)]$ then $N_{r_1}(a)\cap N_{r_2}(b)$ induces a path P_x or P_y. See Figure 5.

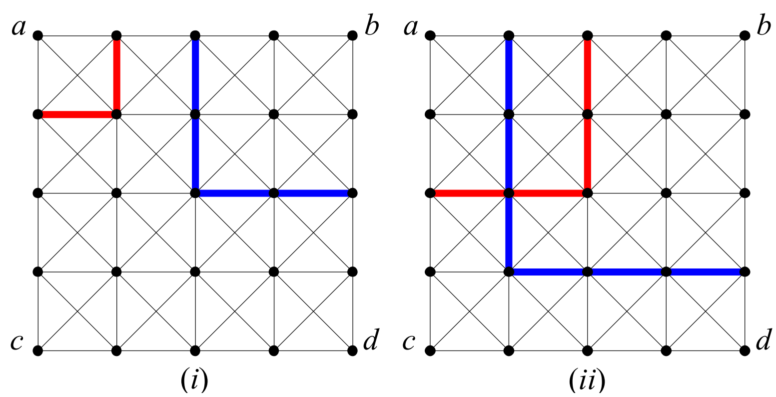

Figure 3: i)$Nr_1(a)\cap Nr_2(b)$; ii) $Nr_2(a)\cap Nr_3(b)$

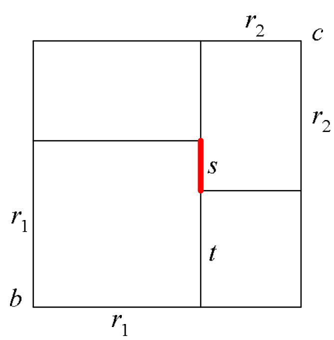

Figure 4: $Nr_1(b)\cap Nr_2(c)$

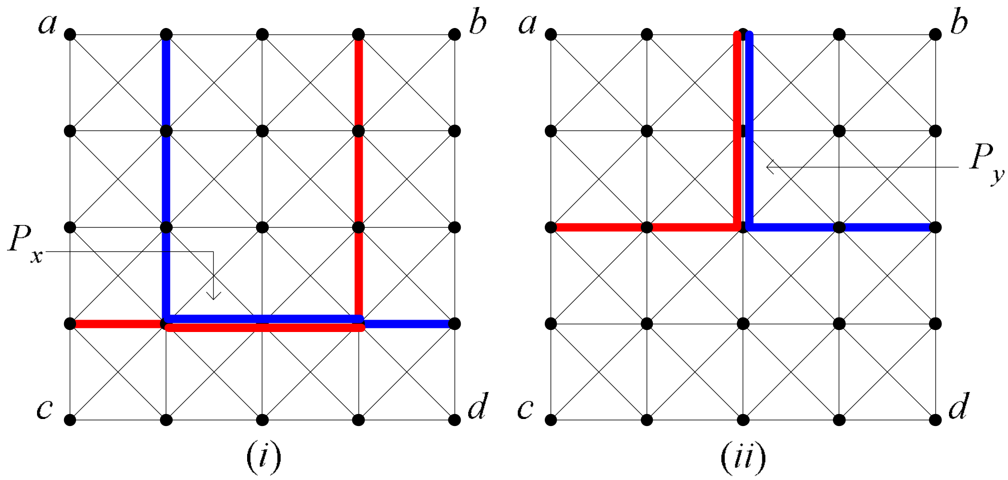

Figure 5: i)Nr₃(a)∩Nr₃(b); ii) Nr₂(a)∩Nr₂(b)

(2) Suppose that $N_{r_1}(b)\cap N_{r_2}(c)$ induces a path say P_y of length $s\geq 1$. See Figure 4. Assume that $r_1=s+t$, where $s\geq 1$. Now $r_1+r_2=m$ and $r_2+t=m$. Hence, $r_1=t$. This gives $s=0$, a contradiction. Similarly $N_{r_1}(b)\cap N_{r_2}(c)$ cannot induce a path P_x of length ≥ 1. Consequently $|N_{r_1}(b)\cap N_{r_2}(c)|\leq 2$. See Figure 6.

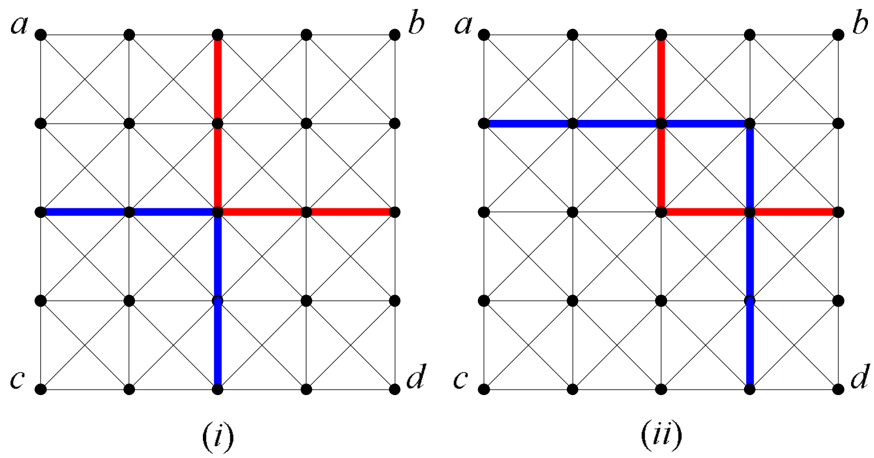

Figure 6: i) Nr₂(b)∩Nr₂(c); ii) Nr₃(b)∩Nr₃(c)

Theorem 1: Let G be an extended mesh $EX(m,m)$. Then $\beta(G)=3$.

Proof: In view of Lemma 1, we need to exhibit a metric basis of cardinality 3. We claim that $\{a,b,c\}$ is a metric basis. Suppose that there are two vertices u and v equidistant from both a and b; in other words suppose that $u,v\in N_{r_1}(a)\cap N_{r_2}(b)$ for some r_1, r_2, $0<r_1,r_2\leq m$. Then by condition (1)

of Lemma 2, u and v induce a path P_x or P_y. But then by condition (2) of Lemma 2, $u, v \notin N_{r_2}(b) \cap N_{r_3}(c)$. Hence if $u \in N_{r_2}(b)$ then $v \notin N_{r_3}(c)$. This means that u and v equidistant from a and b are at unequal distances from c.

3.2 The Enhanced Mesh *EN*(*m*,*n*)

Let a and b denote the vertices of $EN(m,n)$ in (1,1) and $(1,n)$ positions respectively. The vertices of $N_r(a)$ induce a path P in $EN(m,n)$ and this path is in a direction parallel to the diagonal of the $m \times n$ mesh. See Figure 7.

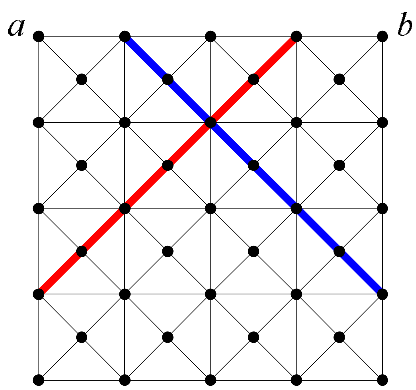

Figure 7: $N_{r_3}(a) \cap N_{r_3}(b)$

Theorem 2: Let G be the Enhanced Mesh $EN(m,n)$. Then $\beta(G)=2$.

Proof: Consider the vertices a and b. It follows from the structure of $EN(m,n)$ that $N_{r_1}(a) \cap N_{r_2}(b)$ is either empty or a singleton. Hence any two vertices are at unequal distances from either a or b.

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