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Inclusions of Measurable Spaces

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ABSTRACT

Let X be a non-empty set, A be a σ - algebra of subsets of X and μ is a positive measure on A so that the triad (X, A, μ) is a measure space. The members of the class A are called measurable sets. A measurable function f on X is said to be in $L^p(\mu)$ if $\int_X |f|^p \, d\mu < \infty$. $L^p(\mu)$ is a normed linear space for $p \geq 1$. $L^\infty(\mu)$ is also a normed linear space. Thus L^p – spaces are normed linear spaces for $1 \leq p \leq \infty$.

In this Paper we investigate the conditions under which L^p - spaces will be contained in L^q - spaces where $0 < p < q$.

Keywords: measure space, normed linear space, σ – algebra.

1. INTRODUCTION

Let X be a non-empty set, A be a σ - algebra of subsets of X and μ is a positive measure on A so that the triad (X, A, μ) is a measure space. The members of the class A are called measurable sets. Let p be such that $0 < p < \infty$. A measurable function f on X is said to be in $L^p(\mu)$ if $\int_X |f|^p \, d\mu < \infty$. If $f \in L^p(\mu)$ and $g = f$ a . e [μ], then $g \in L^p(\mu)$, since the integrals of almost everywhere functions are equal. Therefore we do not distinguish between functions in $L^p(\mu)$ which are a . e [μ].

In other words, the elements of $L^p(\mu)$ are not functions but equivalence classes of functions. With this understanding it is well known that for $p \geq 1$, $L^p(\mu)$ is a normed linear space with L^p – norm defined by $\| f \|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ $for\ any\ f \in L^p(\mu)$. Denote the class of all measurable functions which are bounded almost everywhere on X by $L^\infty(\mu)$.

If we define the L^∞ - norm of $f \epsilon L^\infty(\mu)$ by $\| f \|_\infty = ess.sup\, | f |$, then $L^\infty(\mu)$ is also a normed linear space. Thus the L^p – spaces are defined for every p with $0 < p \le \infty$, but these are normed linear spaces for $1 \le p \le \infty$.

2. PRELIMINARIES

2.1 Lemma: If $L^1(\mu) \cap L^2(\mu)$ then $L^1(\mu) \cup L^\infty(\mu)$.

Proof: Suppose $f \epsilon L^1(\mu)$ and $f \notin L^\infty(\mu)$. We will show that there is a function $h \epsilon L^1(\mu)$ such that $h \notin L^2(\mu) \cup L^\infty(\mu)$.

We may assume that if f is non negative. For, otherwise, we can take the positive part f^+ instead of f . we know $f^+ = (f, 0)$ and that f^+ is non - negative.

Define $g : X \to R$ by

$$g(x) = \begin{cases} (f(x))^{1/2} & for\ f(x) \ge 1 \\ 0 \ for\ 0 \le f(x) < 1 \end{cases}$$

Then $g \notin L^\infty(\mu)$ since $f \notin L^\infty(\mu)$. Also since , $g \le g^2 \le f$, we have $g \epsilon L^1(\mu) \cap L^2(\mu)$. Let $A_n = \{ x : (n-1) \le g(x) < n \}$. Then $\{A_n\}$ is a pair wise disjoint sequence of measurable sets and $\bigcup_{n=1}^{\infty} A_n = X$. If $\mu(A_n) = 0$ for some n, then $\int_{A_n} g\, d\mu = 0$ and therefore we can ignore such A_n's. Thus we assume $C_n = \mu(A_n) > 0$ for every n.

Since $g \epsilon L^1(\mu) \cap L^2(\mu)$ we have

$$\infty > \int_x g\ d\mu = \sum_{n=1}^{\infty} \int_{A_n} g\ d\mu \ge \sum_{n=1}^{\infty} (n-1)\, C_n \quad and$$

$$\infty > \int_x g^2\ d\mu = \sum_{n=1}^{\infty} \int_{A_n} g^2\ d\mu \ge \sum_{n=1}^{\infty} (n-1)^2\, C_n$$

Thus both the series

(2.1.1) $\sum_{n=1}^{\infty}(n-1)C_n$ and $\sum_{n=1}^{\infty}(n-1)^2\, C_n$ are convergent.

But since $\sum_{n=1}^{\infty} C_n \le C_1 + \sum_{n=2}^{\infty}(n-1)C_n$, it follows from (2.1.1) that

(2.1.2) $\sum_{n=1}^{\infty} C_n$ is also convergent.

Again since arithmetic mean of a and b where $a = (n-1)^2 C_n$ and $b = 1/(n-1)^2$, is greater than or equal to their geometric mean, we get the inequality

$$2\sqrt{C_n} \le (n-1)^2\ C_n + 1/_{(n-1)^2}\ for\ all\ n \ge 2.$$ Therefore,

(2.1.3) $2\sum_{n=2}^{\infty}\sqrt{C_n} \le \sum_{n=2}^{\infty}(n-1)^2 C_n + \sum_{n=2}^{\infty} 1/_{(n-1)^2}$

Now (2.1.1) and (2.1.3) imply that

(2.1.4) $\sum_{n=1}^{\infty} C_n$ *converges.*

Define $h : X \to R$ by

$$h(x) = (n-1) + {1}/{\sqrt{C_n}} \quad if \; x \in A_n \,,$$

Then, $\int_X h \; d\mu \;=\; \sum_{n=1}^{\infty} \int_{A_n} h \; d\mu \;=\; \sum_{n=1}^{\infty} \left((n-1) + {1}/{\sqrt{C_n}} \right) \mu \,(A_n)$

$$= \sum_{n=1}^{\infty} (n-1) C_n \;+\; \sum_{n=1}^{\infty} \sqrt{C_n}$$

so that (2.1.1) and (2.1.4) imply that

$\int$ h dμ $< \infty$ proving h $\varepsilon\, L^1(\mu)$. Again,

$$\int_X h^2 \; d\mu \;=\; \sum_{n=1}^{\infty} \int_{A_n} h^2 \, d\mu \;=\; \sum_{n=1}^{\infty} \left((n-1) + {1}/{\sqrt{C_n}} \right)^2 \mu \,(A_n)$$

$= \sum_{n=1}^{\infty} \left\{ (n-1)^2 C_n \;+\; 2\,(n-1)\sqrt{C_n} \;+\; 1 \right\}$, showing $h \notin L^2(\mu)$.

Thus if $f \;\varepsilon\; L^1(\mu)$ and $f \notin L^\infty(\mu)$, there is a $h \;\varepsilon\; L^1(\mu)$ and $h \notin L^2(\mu)$. In other words, $L^1(\mu) \not\subset L^\infty(\mu)$ implies $L^1(\mu) \not\subset L^2(\mu)$, proving the lemma.

3. Main Result

3.1 Theorem: Let (X, $\mathcal{A}$, μ) be a measure space with $\mu(X) = \infty$. For $0 < p < q$, $L^p(\mu) \not\subset L^q(\mu)$ holds iff for any sequence $\{ E_n \}$ of disjoint measurable sets of positive measure, the sequence $\{ \mu(E_n) \}$ is bounded away from zero.

Proof : Suppose $L^p(\mu) \not\subset L^q(\mu)$.

If possible, $\{ E_n \}$ is a sequence of disjoint measurable sets such that $\mu(E_n) > 0$ and $\mu(E_n) \to 0$ as $n \to \infty$. Then for each k, there is a set E_{n_k} *such that* $0 < \mu(E_{n_k}) < \frac{1}{k^3}$.

Defining $h : X \to R$ by

$$h\,(x) \;=\; \begin{cases} k & if \quad x \in E_{n_k} \\ 0 & if \;\; x \notin \bigcup_{k=1}^{\infty} E_{n_k} \end{cases}$$

we find $h \notin L^\infty(\mu)$. But since $\int h \; d\mu \;=\; \sum_{k=1}^{\infty} \int_{E_{n_k}} h \; d\mu \;=\; \sum_{k=1}^{\infty} k\mu(E_{n_k})$

$$\le \; \sum_{k=1}^{\infty} k \, \frac{1}{k^3} \;=\; \sum_{k=1}^{\infty} \frac{1}{k^2} \;<\; \infty\,,$$

It follows that $h \;\varepsilon\; L^1(\mu)$. Thus $h \;\varepsilon\; L^1(\mu)$ and $h \notin L^\infty(\mu)$, so that by Lemma 2.1, $L^1(\mu) \not\subset L^2(\mu)$. That is $L^p(\mu) \not\subset L^q(\mu)$ cannot hold for all p, q with $0 < p < q$ unless the condition holds.

Conversely, suppose (X, $\mathcal{A}$, μ) is a measure space in which the condition of theorem holds. We have to show that $L^p(\mu) \cdot L^q(\mu)$ for all p, q with $0 < p < q$.

Let $f \;\varepsilon\; L^p(\mu)$. If $A_n = \{ x : n < | f(x) | \le n+1 \}$ for $n \ge 0$, then $\{ A_n \}$ is a disjoint sequence of measurable sets. We first note that

(3.3.1) there is a non-negative integer n_0 such that $\mu(A_n) = 0$ for all $n > n_0$.

In fact if (3.3.1) fails we can find an infinite sequence n_k of natural numbers $\mu\ (A_{nk}) > 0$ for each k and then $\int_X |f|^p\ d\mu = \sum_{n=0}^{\infty} \int_{A_n} |f|^p\ d\mu > \sum_{n=0}^{\infty} n^p \mu\ (A_n)$

$$\geq \sum_{k=1}^{\infty} n_k^p\ \mu\ (A_{nk}) = +\infty,$$

a contradiction to the fact that f ε $L^p(\mu)$. Suppose $q > p$, then

(3.3.2) $\quad |f(x)|^q < |f(x)|^p \quad if\ x \in A_0$ and

(3.3.3) $\quad (n+1)^q = (1+1/n)^p\ (n+1)^{q-p}\ n^p$

$$\leq 2^p\ (n_0+1)^{q-p}\ n^p \quad for\ 1 < n < n_0$$

In view of (3.3.1) we have

(3.3.4) $\quad \int_X |f|^q\ d\mu = \int_{A_0} |f|^q\ d\mu + \sum_{n=1}^{n_0} \int_{A_n} |f|^q\ d\mu$

Now using (3.3.2) and (3.3.4), we get

(3.3.5) $\quad \int_X |f|^q\ d\mu \leq \int_{A_n} |f|^p\ d\mu + \sum_{n=1}^{n_0} (n+1)^q\ \mu\ (A_n)$

$$\leq \int_{A_0} |f|^p\ d\mu + 2^p\ (n_0+1)^{q-p} \sum_{n=1}^{n_0} n^p \mu\ (A_n)$$

$$\leq k \left(\int_{A_0} |f|^p\ d\mu + \sum_{n=1}^{n_0} \int_{A_n} |f|^p\ d\mu \right)$$

$= k \int_X |f|^q\ d\mu < \infty$, where $k = \max\{1, 2^p (n_0+1)^{q-p}\}$.

Thus f ε $L^q(\mu)$. Proving $L^p(\mu) \not\subset L^q(\mu)$ for all p, q with $0 < p < q$.

REFERENCES

1. Royden, H.L., Real Analysis, Macmillan, New York (1968).
2. Romero, Juan. L., when is $L^p(\mu)$ contained in $L^q(\mu)$, *Amer. Math. Monthly*, 90, 203 – 206 (1983).
3. Rudin, Walter., Functional Analysis, Mcgraw- Hill (1974).
4. Villani, Alfonso., Another note on the inclusion $L^p(\mu) \not\subset L^q(\mu)$, *Amer. Math. Monthly*, 92, 485-487 (1985).
5. Zaanen, A,C., Integration, North-Holland, Amsterdam, (1967).