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Minimum Risk Estimation of Scale Parameter of Inverse Rayleigh Distribution Under Asymmetric Loss Function

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ABSTRACT

The inverse Rayleigh distribution is one of the most important distributions from lifetime model. This distribution has been used frequently as a model for survival analysis, life testing and industrial reliability problems. In this paper, the minimum risk estimators of its scale have been obtained under Linex loss, precautionary loss, and the other two loss functions. The relative efficiencies of the estimators have been calculated for comparison.

Keywords: Inverse Rayleigh distribution, Maximum likelihood estimator, Risk function, Linex loss, Precautionary loss.

1. INTRODUCTION

The inverse Rayleigh (IR) distribution is considered to be a very useful life time distribution. It has many applications in the area of survival analysis, reliability theory, and life testing study. This model was first introduced by Trayer (1964) and he discussed its applicability to model reliability and survival data sets. Since then this distribution has been applied in many situations involving survival analysis, life testing and industrial reliability problems. Voda (1972) discussed its properties and the maximum likelihood estimator of the scale parameter of this distribution. Gharraph (1993) derived closed form expression of five measures of the parameter of IR distribution. Mukherjee and Maiti (1996) derived percentile estimator of the IR parameter. El-Helbaby and Abd-El-Monem (2005) obtained Bayesian estimators under four loss functions and predictions for IR distribution. Muhammad (2007) discussed some distributional properties and estimated the parameters of IR distribution using lower record values. Inference for IR model has been considered by Howlader *et al*. (2009), Soliman *et al*. (2010) and Dey (2012).

A random variable X is said to follow inverse Rayleigh (IR) distribution if its probability density function (pdf) is given by:

$$f(x;\theta) = \frac{2}{\theta x^3} \exp\left(-\frac{1}{\theta x^2}\right), \quad x > 0. \tag{1.1}$$

where, $\theta > 0$ is the scale parameter.

The corresponding cumulative distribution function (cdf) is given by:

$$F(x;\theta) = \exp\left(-\frac{1}{\theta x^2}\right), \quad x > 0, \theta > 0. \tag{1.2}$$

In this paper, the estimation of the parameter of the IR distribution is considered. Let T be a random variable having the IR distribution, its pdf being defined as;

$$f(t;\theta) = \frac{2}{\theta t^3} \exp\left(-\frac{1}{\theta t^2}\right), \quad t > 0, \theta > 0. \tag{1.3}$$

Let us suppose that there are n persons in the study and everyone is followed to death or failure. Let $t_1, t_2, \ldots\ldots\ldots t_n$ be the exact survival times of the n individuals under investigation. If their survival times follow the IR distribution, the likelihood function is given by,

$$L(\theta; t) = \frac{2^n}{\theta^n} \prod_{i=1}^{n} \frac{1}{t_i^3} e^{-\frac{t}{\theta}} \tag{1.4}$$

where, $t = \sum_{i=1}^{n} \frac{1}{t_i^2}$, is the observation of $T = \sum_{i=1}^{n} \frac{1}{T_i^2}$

The maximum likelihood estimator (MLE) of θ is obtained as;

$$\hat{\theta} = \frac{T}{n} = \frac{\sum_{i=1}^{n} \frac{1}{T_i^2}}{n} \tag{1.5}$$

The distribution of T is found to follow the Gamma distribution $\Gamma(n, \theta^{-1})$.

Since, $\hat{\theta} = \frac{T}{n}$, the pdf of $\hat{\theta}$ is obtained as,

$$f(\hat{\theta}) = \frac{(n/\theta)^n}{\Gamma(n)} \hat{\theta}^{n-1} e^{-n(\frac{\hat{\theta}}{\theta})}, \quad \hat{\theta} > 0. \tag{1.6}$$

2. THE MINIMUM EXPECTED LOSS (RISK) ESTIMATOR UNDER LINEX LOSS FUNCTION(II)

Varian (1975) introduced the asymmetric loss function known as LINEX (Linear Exponential) loss function defined as,

$$L(\delta) = b\left(e^{a\delta} - a\delta - 1\right), \quad a \neq 0, b > 0, \tag{2.1}$$

where a and b are the shape and scale parameters of the loss function, respectively. Basu and Ebrahimi (1991) considered this loss function with $\delta = \frac{\hat{\theta}}{\theta} - 1$, to study reliability estimation in exponential distribution. If we define

$$\delta = \hat{\theta} - \theta, \tag{2.2}$$

Then,

$$L(\delta) = b[e^{a(\hat{\theta}-\theta)} - a(\hat{\theta} - \theta) - 1] \tag{2.3}$$

where $\hat{\theta}$ is the MLE of θ with the pdf given by (1.6).

Now, let us define the estimator θ^* as follows

$$\theta^* = M\hat{\theta} \tag{2.4}$$

where M is a scalor.
For this estimator the Linex loss function takes the form

$$L(\delta^*) = b[e^{a\delta^*} - a\delta^* - 1],\ a \neq 0, b > 0; \tag{2.5}$$

where $\delta^* = (\theta^* - \theta)$. Hence, the loss function (2.5) becomes

$$L(\delta^*) = b[e^{a(\theta^*-\theta)} - a(\theta^* - \theta) - 1],\quad a \neq 0, b > 0 \tag{2.6}$$

Let us consider the two estimators as follows;
(i) MLE $\hat{\theta}$; and
(ii) θ^* = M$\hat{\theta}$;
where M is so chosen that the expected loss (the risk) is minimum.
The risk function of MLE $\hat{\theta}$ under Linex loss function denoted by $R_L(\hat{\theta})$, is given by

$$R_L(\hat{\theta}) = \mathrm{E}\,[L(\delta)] = \int_0^\infty L(\delta)\, f(\hat{\theta})\, d\hat{\theta} \tag{2.7}$$

Now substituting the value of L (δ) and $f(\hat{\theta})$ from (2.3) and (1.6) in (2.7), we get;

$$R_L(\hat{\theta}) = \frac{(n/\theta)^n}{\Gamma(n)} \int_0^\infty b[e^{a(\hat{\theta}-\theta)} - a(\hat{\theta} - \theta) - 1]\, \hat{\theta}^{n-1}\, e^{-n(\frac{\hat{\theta}}{\theta})}\, d\hat{\theta}$$

or, $$R_L(\hat{\theta}) = \mathrm{b}(n/\theta)^n\, [e^{-a\theta}(\tfrac{n}{\theta} - a)^{-n} - an(\tfrac{n}{\theta})^{-(n+1)} + (a\theta - 1)(\tfrac{n}{\theta})^{-n}] \tag{2.8}$$

Similarly, the risk function under Linex loss function for the estimator
θ^* = M$\hat{\theta}$ is given by,

$$R_L(\theta^*) = \mathrm{E}\,[L(\delta^*)] = \int_0^\infty L(\delta^*)\, f(\hat{\theta})\, d\hat{\theta} \tag{2.9}$$

Now substituting the value of $L(\delta^*)$ and $f(\hat{\theta})$ from (2.6) and (1.6) in (2.9), we get,

$$R_L(\theta^*) = \mathrm{b}(n/\theta)^n[e^{-a\theta}(\tfrac{n}{\theta} - aM)^{-n} - aMn(\tfrac{n}{\theta})^{-(n+1)} + (a\theta - 1)(\tfrac{n}{\theta})^{-n}] \tag{2.10}$$

In order to minimize this risk with respect to M, we have,

$$\frac{d}{dM} R_L(\theta^*) = \mathrm{b}(n/\theta)^n\, [e^{-a\theta}(\tfrac{n}{\theta} - aM)^{-n-1}an - an(\tfrac{n}{\theta})^{-(n+1)}] \tag{2.11}$$

and

$$\frac{d^2}{dM^2} R_L(\theta^*) = \mathrm{b}(n/\theta)^n\, [e^{-a\theta}\left(\frac{n}{\theta} - aM\right)^{-n-2} an(n+1)a] \tag{2.12}$$

Since, $\frac{d^2}{dM^2} R_L(\theta^*) > 0$ if, $\frac{n}{\theta} - aM > 0$, therefore M is minimum if
$\frac{d}{dM} R_L(\theta^*) = 0$, which on using (2.11) gives

$$\mathrm{M} = (\tfrac{n}{a\theta})\, [1 - e^{-\frac{a\theta}{n+1}}] \tag{2.13}$$

Since M depends upon the parameter θ to be estimated, the true value of M cannot be obtained. The approximate value of M may be obtained by replacing θ by $\hat{\theta}$. If we replace θ by $\hat{\theta}$ the estimator of M becomes,

$$\hat{M} = (\tfrac{n}{a\hat{\theta}})\, [1 - e^{-\frac{a\hat{\theta}}{n+1}}] \tag{2.14}$$

We want to obtain an approximate value of M which does not depend on the parameter. Hence,

$$\mathrm{M} = (\tfrac{n}{a\theta})\, [1 - e^{-\frac{a\theta}{n+1}}] = (\tfrac{n}{a\theta})\, [\,\frac{a\theta}{n+1} - \frac{1}{2}(\frac{a\theta}{n+1})^2 + \frac{1}{6}\left(\frac{a\theta}{n+1}\right)^3 - \cdots \ldots \ldots \ldots \ldots\,]$$

Thus, by neglecting higher powers of (n+1) in the denominator, we have the approximate value of M as,

$$\widehat{M} \cong \frac{n}{n+1} \tag{2.15}$$

Thus, we get the approximate estimator θ^* as,

$$\theta^* = \widehat{M}\widehat{\theta} = \frac{n}{n+1}\widehat{\theta} \tag{2.16}$$

The relative efficiency of the estimators with respect to the estimator $\widehat{\theta}$ is defined as

$$\text{Rel. eff. } (\theta^*/\widehat{\theta}) = \frac{R_L(\widehat{\theta})}{R_L(\theta^*)}$$

$$\text{Rel. eff. } (\theta^*/\widehat{\theta}) = \left[\frac{e^{-a\theta}(\frac{n}{\theta}-a)^{-n} - an(\frac{n}{\theta})^{-(n+1)} + (a\theta-1)(\frac{n}{\theta})^{-n}}{e^{-a\theta}(\frac{n}{\theta}-aM)^{-n} - aMn(\frac{n}{\theta})^{-(n+1)} + (a\theta-1)(\frac{n}{\theta})^{-n}}\right] \tag{2.17}$$

The values of the relative efficiency may be calculated for different values of r and θ. The relative efficiency of the MELO estimator θ^* with respect to the MLE $\widehat{\theta}$, under Linex loss function has been calculated for some values of n and θ, in the table below:

Table 1.

θ	N	Rel.eff.
0.025	5	1.202842
0.05	10	1.102675
0.075	15	1.069286
0.1	20	1.052591
0.125	25	1.042575
0.15	30	1.035897
0.175	35	1.031127
0.2	40	1.02755
0.225	45	1.024767

3. THE MINIMUM EXPECTED LOSS (RISK) ESTIMATOR UNDER PRECAUTIONARY LOSS FUNCTION

Norstrom (1996) introduced the precautionary loss function which is asymmetric in nature and also presented a general class of precautionary loss functions with squared error loss function as a special case. A very useful and simple asymmetric precautionary loss function is,

$$L(\delta) = \frac{(\theta-\widehat{\theta})^2}{\widehat{\theta}} \tag{3.1}$$

The risk function of MLE $\hat{\theta}$ under precautionary function, denoted by $R_p(\hat{\theta})$ is given by,

$$R_p(\hat{\theta}) = E[L(\delta)] = \int_0^\infty L(\delta)\, f(\hat{\theta})\, d\hat{\theta} \tag{3.2}$$

Now substituting the value of L (δ) and $f(\hat{\theta})$ from (3.1) and (1.6) in (3.2), we get;

$$R_p(\hat{\theta}) = \int_0^\infty \frac{(\theta-\hat{\theta})^2}{\hat{\theta}} \frac{(n/\theta)^n}{\Gamma(n)} \hat{\theta}^{n-1} e^{-n(\frac{\hat{\theta}}{\theta})} d\widehat{\theta} \tag{3.3}$$

This gives us,

$$R_p(\hat{\theta}) = \frac{\theta}{n-1} \tag{3.4}$$

Similarly, the risk function under precautionary loss function for the estimator $\theta^* = M\hat{\theta}$ is given by,

$$R_p(\theta^*) = E\,[L(\delta^*)] = \int_0^\infty L(\delta^*)\, f(\hat{\theta})\, d\hat{\theta} \tag{3.5}$$

where, $\theta^* = M\hat{\theta}$ and $\delta^* = (\theta^* - \theta)$.

Therefore,

$$R_p(\theta^*) = \int_0^\infty \frac{(\theta - M\hat{\theta})^2}{M\hat{\theta}} \frac{(n/\theta)^n}{\Gamma(n)}\, \hat{\theta}^{n-1}\, e^{-n(\frac{\hat{\theta}}{\theta})}\, d\hat{\theta} \tag{3.5}$$

Thus, we get

$R_p(\theta^*) = [\frac{\theta n}{(n-1)M} + M\theta - 2\theta]$ (3.6)

In order to minimize this risk with respect to M, we have

$$\frac{d}{dM} R_P(\theta^*) = -\left(\frac{n\theta}{n-1}\right)\frac{1}{M^2} + \theta; \text{ and } \frac{d^2}{dM^2} R_P(\theta^*) = \left(\frac{n\theta}{n-1}\right)\frac{2}{M^3} > 0 \tag{3.7}$$

The value of M that minimizes $R_P(\theta^*)$ is found by solving the equation,

$\frac{d}{dM} R_P(\theta^*) = -\left(\frac{n\theta}{n-1}\right)\frac{1}{M^2} + \theta = 0$, which gives the value of M to be,

$$M = \sqrt{\frac{n}{n-1}} \tag{3.8}$$

Here, M is dependent only on n and does not depend on any unknown parameter. Now, θ^* becomes

$$\theta^* = \sqrt{\frac{n}{n-1}}\,\hat{\theta} \tag{3.9}$$

The relative efficiency of the estimator θ^* with respect to the estimator $\hat{\theta}$ is obtained as,

$$\text{Rel. eff. } (\theta^*/\hat{\theta}) = \frac{R_P(\hat{\theta})}{R_P(\theta^*)} = \frac{\theta/(n-1)}{[\left(\frac{n}{n-1}\right)\frac{\theta}{M} + M\theta - 2\theta]} = \frac{1}{2(\sqrt{n(n-1)} - (n-1))} \tag{3.10}$$

In order to have Rel. eff. $(\theta^*/\hat{\theta}) \geq 1$, we have that,

$$\sqrt{n(n-1)} - (n-1) \leq \frac{1}{2} \tag{3.11}$$

which is always true. Thus, θ^* always dominates $\hat{\theta}$.

4. THE MINIMUM EXPECTED LOSS (RISK) ESTIMATOR UNDER AN OTHER LOSS FUNCTION(I)

A useful loss function is given by,

$$L(\delta) = \left(\frac{\hat{\theta}}{\theta} - 1\right)^2 \tag{4.1}$$

The risk function of MLE $\hat{\theta}$ under the loss function I denoted by $R_1(\hat{\theta})$ is given by,

$$R_1(\hat{\theta}) = E\,[L(\delta)] = \int_0^\infty L(\delta)\, f(\hat{\theta})\, d\hat{\theta} = \int_0^\infty \left(\frac{\hat{\theta}}{\theta} - 1\right)^2 \frac{(n/\theta)^n}{\Gamma(n)}\, \hat{\theta}^{n-1}\, e^{-n(\frac{\hat{\theta}}{\theta})}\, d\hat{\theta} \tag{4.2}$$

Then, we have

$R_1(\hat{\theta}) = \frac{1}{n}$ (4.3)

Similarly, the risk function under the loss function I for the estimator θ^* is given by,

$R_1(\theta^*)$ = E $[L(\delta^*)] = \int_0^\infty L(\delta^*)\, f(\hat{\theta})\, d\hat{\theta}$ (4.4)

where, $\theta^* = M\hat{\theta}$ and $\delta^* = (\theta^* - \theta)$.

Thus,

$$R_1(\theta^*) = \int_0^\infty \left(\frac{M^2\hat{\theta}^2}{\theta^2} - \frac{2M\hat{\theta}}{\theta} + 1\right) \frac{(n/\theta)^n}{\Gamma(n)}\, \hat{\theta}^{n-1}\, e^{-n(\frac{\hat{\theta}}{\theta})} d\hat{\theta}$$

This gives,

$R_1(\theta^*) = [\left(\frac{n+1}{n}\right) M^2 - 2M + 1]$ (4.5)

In order to minimize this risk with respect to M, we have

$\frac{d}{dM}\, R_1(\theta^*) = 0$, which gives

$M = \frac{n}{n+1}$ (4.6)

Since,

$\frac{d^2}{dM^2}\, R_1(\theta^*) = \frac{2(n+1)}{n} > 0$,

the value of M given by equation (4.6) minimizes the value of $R_1(\theta^*)$. Thus, the estimator θ^* becomes

$\theta^* = \frac{n}{n+1}\hat{\theta}$ (4.7)

The relative efficiency of the estimator θ^* with respect to the estimator $\hat{\theta}$ is obtained as,

Rel. eff. $(\theta^*/\hat{\theta}) = \frac{R_1(\hat{\theta})}{R_1(\theta^*)} = \frac{n+1}{n} > 1$. (4.8)

(4.8) is true uniformly in n and θ. Thus, θ^* uniformly dominates $\hat{\theta}$. Therefore, θ^* is always preferable to $\hat{\theta}$ when the loss function (4.1) is considered.

5. THE MINIMUM EXPECTED LOSS (RISK) ESTIMATOR UNDER AN OTHER LOSS FUNCTION(II)

Let us consider another loss function II, defined by

$L(\delta) = (\frac{\theta}{\hat{\theta}} - 1)^2$ (5.1)

The risk function of MLE $\hat{\theta}$ under the loss function II denoted by $R_2(\hat{\theta})$ is given by,

$R_2(\hat{\theta})$ = E $[L(\delta)] = \int_0^\infty L(\delta)\, f(\hat{\theta})\, d\hat{\theta}$

Or, $R_2(\hat{\theta}) = \int_0^\infty (\frac{\theta}{\hat{\theta}} - 1)^2 \frac{(n/\theta)^n}{\Gamma(n)}\, \hat{\theta}^{n-1}\, e^{-n(\frac{\hat{\theta}}{\theta})}\, d\hat{\theta}$ (5.2)

Then, we have

$R_2(\hat{\theta}) = \frac{n^2}{(n-1)(n-2)} - \frac{2n}{n-1} + 1$ (5.2)

Similarly, the risk function under the loss function II for the estimator θ^* is given by,

$R_2(\theta^*)$ = E $[L(\delta^*)] = \int_0^\infty L(\delta^*)\, f(\hat{\theta})\, d\hat{\theta}$ (5.3)

where, θ^* = M$\hat{\theta}$ and $\delta^* = (\theta^* - \theta)$.
Hence,

$$R_2(\theta^*) = \int_0^\infty \left(\frac{M^2\theta^2}{\hat{\theta}} - \frac{2M\theta}{\hat{\theta}} + 1\right) \frac{(n/\theta)^n}{\Gamma(n)}\, \hat{\theta}^{n-1}\, e^{-n(\frac{\hat{\theta}}{\theta})} d\hat{\theta} \qquad (5.4)$$

This gives,

$$R_2(\theta^*) = \frac{n^2M^2}{(n-1)(n-2)} - \frac{2nM}{n-1} + 1 \qquad (5.5)$$

In order to minimize this risk with respect to M, we have
$\frac{d}{dM} R_2(\theta^*) = 0$, which gives

$$M = \frac{n-2}{n} \qquad (5.6)$$

and since,
$\frac{d^2}{dM^2} R_2(\theta^*) = \frac{2\,n^2}{(n-1)(n-2)} > 0$, the value of M given by equation (5.6) minimizes the value of $R_2(\theta^*)$. Thus, the estimator θ^* becomes

$$\theta^* = \left(\frac{n-2}{n}\right) \hat{\theta} \qquad (5.7)$$

The relative efficiency of the estimator θ^* with respect to the estimator $\hat{\theta}$ is obtained as,

$$\text{Rel. eff. } (\theta^*/\hat{\theta}) = \frac{R_2(\hat{\theta})}{R_2(\theta^*)} = \frac{n+2}{n-1} > 1. \qquad (5.8)$$

The inequality (5.8) is always true uniformly in r and θ. Thus, θ^* uniformly dominates $\hat{\theta}$. Therefore, θ^* is always preferable to $\hat{\theta}$ when the loss function (5.1) is considered.

6. CONCLUSION

In this paper, we have obtained the minimum expected loss (MELO) estimator of the scale parameter of IR distribution under different asymmetric loss functions. Table 1 shows the relative efficiencies of the MELO estimator θ^* with respect to the MLE estimator $\hat{\theta}$ for different values of θ and sample size n, for the Linex loss function. It follows from the Table 1. that θ^* is always better than $\hat{\theta}$ for all tabulated values of θ and sample size n, when Linex loss function is considered as the loss criterion. From equations (3.10), (4.8), and (5.8) it is clear that θ^* uniformly dominates $\hat{\theta}$. Therefore, it follows that MELO estimator θ^* is preferable to MLE $\hat{\theta}$, under precautionary and the other two asymmetric loss functions.

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