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On the Vanishing of Vector Fields Associated with Group Actions

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ABSTRACT

In this paper we have seen that the group action on manifolds will set the dynamics and the system under such action have some important implication. We have seen that the circle groups acting on low dimension manifolds have given rise to some nice results.

Keywords: circle groups acting, manifolds, low dimension.

1. INTRODUCTION

Stochasticity has entered the realm of differentiable geometry in a natural manner. The fact that a d-dimensional real Euclidean space providing such an ambience enabled us to extend these ideas tools on smooth manifolds as well. One of the problems associated with the configuration space is its state which is regarded as dynamical systems. Its structural probability is one of the important and remained as an open problem. In this note we intend to investigate problems arising from random environment. The underlying space which is a smooth manifold of dimensional n, admitting probability measure would enable us to formulate many problems associated with the configuration space of the manifold. One such problem is to study the long time behavior of the dynamical system under evolution. In this exposition the underlying manifold is smooth and finite dimensional. The dynamics of the system is realized in terms of group action on manifold. Section 2 deals with the frame work and also some pre requisites are covered. Section 3 deals with the vanishing of vector fields as a result of such a group action. We have made standard references for the stochastic related idea and group action[1,2,3,4]. At the end we have come with our results and stated them in the form of propositions. Elsewhere we have continued our investigations with group actions on smooth manifolds.

2. PROBABILITY MEASURE UNDER EUCLIDEAN FRAME WORK

In this section we present the notion of probability measure on $\mathbb{R}^d$. This would enable us discuss stochastic process on them. Among them markov processes are well understood and help us to understand the evolution of the dynamical system. This evolution process is in fact random. We mean, if X_t denotes the state of the space of the instance t, then the state of the system X_{t+s} of the instance t + s is not known well before or in advance. By means of transition probabilities we can define stochastically the evolution process.

Thus, If $\{ X_t : X_t \in \mathbb{R}^d, t \in \mathbb{R}\}$ constitutes the configuration space then each X_t is the state of the configuration at instance t.

If d=1, then $\{X_t : X_t \in \mathbb{R}^d, t \in \mathbb{R}\}$ are the points on the real line. Thus we have the pairs (t , X_t) for each t$\in \mathbb{R}$.

Clearly we see that the future instance and associated configuration of the system could only $\mathcal{B}$ predicted by assigning probabilities to these transitions of the evolving system. Naturally we see a group action. This group action will be discussed later.

To this end we consider the probability measure space ($\mathbb{R}^d$,$\mathcal{B}, P$) and if the members of $\mathcal{B}$ which are a Borel sets. Thus as subsets of $\mathbb{R}^d$ and Borel measurable the transition probabilities p (t, x, A) can be identified with $X_t \in \mathbb{R}^d$, A as the event set for which the outcome X_t would be realized at instance t with probability p. For each X_t, we have a transition probability p(t, x, A).

On $\mathbb{R}$, the real line.

We consider a set S=$\{X_t$: t $\in \mathbb{R}$, X$\in \mathbb{R}\}$ (usually t is a time parameter) and the map φ_t : S$\rightarrow S$ and tha defining equation is

$X_s \longmapsto \varphi_t(X_s)$ Where $\varphi_t(X_s)= X_{t+s}$

In general φ_t: S$\rightarrow S$ where S $\subseteq \mathbb{R}^d$ may be defined in the same manner to give higher dimension dynamical system. The above discussion culminates into the following propositions.

1 Proposition: In $\mathbb{R} \times \mathbb{R}^d$ the map φ: $\mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\varphi(t, x)= \varphi_t(x)$

2 Proposition: In $\mathbb{R}^d$,d$\geq$ 1 and if S $\subseteq \mathbb{R}^d$ then for the pair of (t , X_t), t $\in \mathbb{R}$, $X_t \in \mathbb{R}^d$ determine the subsets of $\mathbb{R}^d$. Then there is a family of continuous maps φ_t: S$\rightarrow S$ defined by $\varphi_t(X_s)= X_{t+s}$.

The above proposition gives a very nice picture of a dynamical system and $\{\varphi_t$: t $\in \mathbb{R}\}$ constitutes a one parameter family of curves in $\mathbb{R}^d$. For this family of curves we defined a product (composition) which is closed and satisfies group properties. If φ_t and φ_s are two maps then their product defined by

$\varphi_t \circ \varphi_s$=$\varphi_{t+s}$. In view of proposition 1 the one parameter family of map satisfies the group properties.

The above propositions clearly depict the group action on $\mathbb{R}^d$. We give a formal definition of a group action on a manifold and see that group actions set the dynamics are motion of the points in the manifold.

Definition: Let G be group (finite or otherwise, abelian or non-abelian) and M be a smooth n manifold. A group action G on M is a smooth map

$\varphi: G \times M \to M$ defined by (g, x)$\mapsto \varphi(g, x) = gx$.

The group G is said to act on M on its left (similarly group action on the right is defined by xg).

A G -space or a G manifold is a manifold M together with a group action on M. In the above proposition we have seen the group of real's $\mathbb{R}$ acting on the manifold $\mathbb{R}^d$. We will confined to the left action of the group on the manifold.

Definition: Let M be a manifold which is smooth and finite dimensional. Let x$\in M$ and G act on M. Then the orbit through x in M is the set $G_x = \{g_x :$ g$\in G\}$.The action is called transitive if M is one orbit for all x, y $\in M$ there is some g in G such that gx=y

In the next section we deal with circle group action on manifolds for the vanishing if vector field.

3. VANISHING OF VECTOR FIELD ARISING OUT OF GROUP ACTION

Let M be a smooth n dimensional manifold and compact. Let G be group acting on it (left group action). Then we have seen that for each real t φ_t : M$\to M$ is a local diffeomorphism given by φ_t(x)=$\varphi(t, x)$. We have also defined orbit in M under group action.

To this end, let $\gamma: I \to M$ *be* a simple oriented closed curve in M where I is an interval containing 0. i.e, I=$[-\varepsilon, \varepsilon]$, $\gamma(-\varepsilon)$= $\gamma(\varepsilon)$. Let x$\in M$ and $\gamma(0)$=x. It means γ passes through x in M. Notice that this is a natural group action on M by the subgroup of $\mathbb{R}$. A well known result in geometry associated with it is the one parameter group of transformations and they generate vector fields.[3 Boothby]. From the classification themes, we know that a 2-dimensional closed on oriented surfaces is either homeomorphic to a 2 sphere S^2 or to a 2 sphere with k handles. If M is a 2 dimensional closed oriented manifold then it is homeomorphic to a 2 sphere or 2 sphere with k handles.

Example 3.1: (Circle group S^1) The circle group is a 1-dimensional smooth manifolds. There are two descriptions to it. One analytic and the other algebraic. As a geometric entity it can be defined as a subset of $\mathbb{R}^2$ consisting of the order pairs (x, y) such that x^2+y^2=1. Thus the origin (0, 0) will constitute the center for all such order pair of real numbers with fixed distance equal to 1which constitutes the radius of the circle. from the centre. As a manifold it is one dimensional admits a smooth differential structure[3,4].

Example 3.2: For the two dimensional case the sphere S^2 given by the set of all 3 tuples (x, y, z) in $\mathbb{R}^3$ such that x^2+$y^2 + z^2$=1. The differential structure is a smooth 2-dimensional as on S^2. In this case the atlas is also smooth.

Example 3.3: The Cartesian product $S^1 \times \mathbb{R}$ is a two dimensional cylindrical surface. It is obtained by generating the circle about z axis $\mathbb{R}^3$. This is an example of a 2-Dimensional manifold. In place of $\mathbb{R}$ consider a closed unit interval [0, 1], then we get a cylinder of height 1.

4. THE CIRCLE $group\ S^1$ Acting On Manifold

In case of sphere we rotate the circles in the z-axis and notice as the radius of the circle varies from 0 to 1 the surface it generate is a sphere. In other words there is a group action on S^2, combining with the points of circle with the points of sphere.

Imagine a closed path on S^2 containing gp. S^2 is a simple closed surface it is dense with such closed curves.

If $\{\gamma: \gamma \in$ is a simple closed curves in $S^2\}$ coordinality $\{ \gamma: \gamma \in$ is a simple closed curves in $S^2\}$ is a continuous. Therefore for each (g, p)$\in S^1 \times S^2$ gp$\in \gamma_{S^2}$ or gp$\in {\gamma^c}_{S^2}$. Let S^1 act on M, where M$\simeq S^2$. If $\gamma: I \to M,$ is a smooth closed curves in M and $\gamma_{S^2} : I \to S^2$ in S^2 where I= (-ε, ε), $\varepsilon > o,$ containig o. Let γ_M(o)=x then γ_M passes through x in M. So, the map $\mathbb{R} \times M \to M$ where $<\mathbb{R}, +>$ is the additive abelian group is an action M (left group action) i.e, (r, x)$\to \varphi(r, x)$=r.x, if M= $\mathbb{R}^2$, then (r, x) $\longmapsto \varphi(r, x) = rx$.

$\{S^1_\theta: \theta \in \mathbb{R}\}$ generates the cylindrical above the z-axis since M$\simeq S^2$ S^2 being compact we will examine the circle group S^1 on S^2.

So m considered with group element g of G, moves to gm, further element of M, denoting it by m, we examine the circle group S^1 action on S^2.

Proposition 1: If $\{S^1_\theta:\theta\epsilon\mathbb{R}\}$ is a family of circle in $\mathbb{R}^3$, then S^1 action on $S^1 \times \mathbb{R}$ which is a cylinder is generated by $(\theta, x) \to \theta x$ [where $S^1 \times \mathbb{R}$ is the infinite right circular cylinder]

Proposition 2: $\{S^{1/P}$, P$\in S^2\}$ generate S^2 by the circle group action(left group action).

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