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On a Hsu-unified Structure Manifold with a Recurrent Metric Connection

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ABSTRACT

In the present paper, we have defined a Hsu-unified structure manifold and a Hsu-Kahler manifold and studied some properties of the recurrent metric connection in a Hsu-unified structure Manifold. We have also studied the curvature tensor of Hsu-Kahler manifold with respect to the recurrent metric connection.

Keywords: Semi-symmetric recurrent metric connection, Generalized co-symplectic manifold, Generalized quasi-Sasakian manifold.

1. INTRODUCTION

Recurrent metric connection was introduced and studied by Liang,Y[2] in 1994. Several properties of semi-symmetric metric and non-metric connections on a differentiable manifold have been studied by Yano and Imai[5], Agashe and Chafle[1] and many others. In the present paper, we have studied some properties of the semi-symmetric non metric connection on a Hsu-unified structure manifold. It has been shown that the Nijenhuis tensor with respect to semi- symmetric non-metric connection and with respect to Riemannian connection coincide in the Hsu-unified structure manifold but in the Hsu- Kahler manifold Nijenhuis tensor with respect to semi-symmetric metric connection vanishes identically i.e. a Hsu- Kahler manifold is integrable.

2. PRELIMINARIES

An even dimensional differentiable manifold $M_n(n = 2m)$, there exists a vector valued linear function $\emptyset$ of differentiability class C^∞ satisfying

$$\emptyset^2 X = a^r X \qquad (2.1)$$

for arbitrary vector field X . Also there exists a Riemannian metric g, such that

$$g(\bar{X},\bar{Y}) = a^r g(X,Y) \tag{2.2}$$

where $\bar{X} = \emptyset X$, $0 \leq r \leq n$ and a is real or complex number.

Then M_n is said to be a Hsu-unified structure manifold.

Let us define a 2-form F in M_n

$$\text{F(X,Y)} = g(\bar{X},\bar{Y}) = g(X,\bar{Y}) \tag{2.3}$$

Then it is clear that the 2-form F satisfies

$$F(\bar{X},\bar{Y}) = a^r g(X,Y) \tag{2.4}$$

From equation (2.3), we have the following properties

$$F(\bar{X},Y) = a^r g(X,Y) \tag{2.5}$$

$$F(X,Y) = F(Y,X) \tag{2.6}$$

Equation (2.6) shows that the 2-form F is symmetric in M_n.

If the Hsu-unified structure manifold M_n satisfies a condition

$$D_X\bar{Y} = \overline{D_X Y} \Leftrightarrow \overline{D_X\bar{Y}} = a^r(D_X Y) \tag{2.8}$$

3. SEMI SYMMETRIC RECURRENT METRIC CONNECTION

A linear connection B defined as[4]

$$B_X Y = D_X Y - \eta(X,Y) \tag{3.1}$$

For arbitrary vector fields X and Y is said to be a semi –symmetric recurrent metric connection if the torsion tensor S of the connection B and the metric tensor g are given by

$$S(X,Y) = \eta(Y)X - -\eta(X)Y \tag{3.2}$$

And

$$(B_X g)(Y,Z) = 2\eta(X)g(Y,Z) \tag{3.3}$$

where η is one form associated with the vector field ξ such that

$$\eta(X) = g(X,\xi) \tag{3.4}$$

Putting equation (3.1) as.

$$B_X Y = D_X Y + P(X,Y) \tag{3.5}$$

Then

$$P(X,Y) = -\,\eta(X)Y \tag{3.6}$$

Let us define

$$P(X,Y,Z) = -\,g(P(X,Y),Z) \tag{3.7}$$

then by virtue of the equation (3.6), equation (3.7) becomes

$$P(X,Y,Z) = -\,\eta(X)g(Y,Z) \tag{3.8}$$

4. EXISTENCE OF SEMI SYMMETRIC RECURRENT METRIC CONNECTION

A linear connection B defined as[4]

$$B_X Y = D_X Y + P(X,Y) \tag{4.1}$$

where P is a tensor of type (1,2) for arbitrary vector fields X and Y is said to be a semi-symmetric recurrent metric connection if the torsion tensor S of the connection B and the metric tensor g are given by

$$S(X,Y) = P(X,Y) - P(Y,X) \tag{4.2}$$

And

$$G(X,Y,Z) = (B_X g)(Y,Z) \tag{4.3}$$

$$g(P(X,Y),Z) + g(P(X,Z),Y) = -G(X,Y,Z) \tag{4.4}$$

From equations (4.1), (4.2) and (4.4), we have

$$g(S(X,Y),Z) + g(S(Z,X),Y) + g(S(Z,Y),X)$$
$$= g(P(X,Y),Z) - g(P(Y,X),Z) + g(P(Z,X),Y) - g(P(X,Z),Y) + g(P(Z,Y),X) - g(P(Y,Z),X)$$
$$= 2g(P(X,Y),Z) + 2\eta(X)g(Y,Z) + 2\eta(Y)g(X,Z) - 2\eta(Z)g(X,Y)$$

$$P(X,Y) = \frac{1}{2}\left\{S(X,Y) + \check{S}(X,Y) + \check{S}(Y,X)\right\} - \eta(X)Y - \eta(Y)X = g(X,Y) \tag{4.5}$$

where

$$g\left(\check{S}(X,Y),Z\right) = g(S(Z,X)Y). \tag{4.6}$$

From equations (4.6) and (3.4), we get

$$\check{S}(X,Y) = \eta(X)Y - g(X,Y)\xi \tag{4.7}$$

Then in view of equations (4.5) and (4.7), we get

$$P(X,Y) = -\eta(X)Y \tag{4.8}$$

Which implies

$$B_X Y = D_X Y - \eta(X,Y) \tag{4.9}$$

5. HSU-UNIFIED STRUCTURE MANIFOLD EQUIPPED WITH SEMI SYMMETRIC RECURRENT METRIC CONNECTION

In this section, we have the following theorems:

Theorem 5.1. For a Hsu-unified structure manifold M_n equipped with a semi symmetric recurrent metric connection , the following results hold good

$$\left.\begin{aligned}
&(i)\ P\left(\bar{X},\bar{\bar{Y}}\right) = a^r P(\bar{X},Y)\\
&(ii)\ P\left(\bar{\bar{X}},\bar{Y}\right) = a^r P(X,\bar{Y})\\
&(iii)\ P\left(\bar{\bar{X}},Y\right) = P\left(X,\bar{\bar{Y}}\right) = a^r P(X,Y)\\
&(iv)\ P\left(X,\bar{\bar{Y}}\right) = P\left(Y,\bar{\bar{X}}\right)\ iff\ \eta(X)Y = \eta(Y)X\\
&(v)\ P(X,\bar{Y}) = P(Y,\bar{X})\quad iff\ \eta(X)\bar{Y} = \eta(\bar{X})Y
\end{aligned}\right\} \tag{5.1}$$

Proof : From the equations (3.6) and (2.1), we have

$$\left.\begin{aligned}
&(i)\ P\left(\bar{X},\bar{\bar{Y}}\right)Y = -a^r\eta(\bar{X})Y\\
&(ii) P\left(\bar{\bar{X}},\bar{Y}\right) = -a^r\eta(X)\bar{Y}\\
&(iii)\ \ P\left(\bar{\bar{X}},Y\right) = P\left(X,\bar{\bar{Y}}\right) = -a^r\eta(X)Y
\end{aligned}\right\} \tag{5.2}$$

Clearly equations (5.1) (i) , (ii)and (iii) follows from equations (5.2) (i), (5.2) (ii) and (5.2) (iii). Interchanging X and Y in equation (3.6), we get

$$P(Y,X) = -\eta(Y)X \tag{5.3}$$

Barring X in equation (5.3) and using equation (2.1), we have

$$P(Y,\bar{\bar{X}}) = -a^r\eta(Y)X \tag{5.4}$$

Now the result of (5.1) (iv) follows from equations (5.2) (iii) and (5.4). Again barring X in equation (5.3),we have

$$P(Y,\bar{X}) = -\eta(Y)\bar{X} \tag{5.5}$$

From the equations (3.6) and (2.1), we have

$$P(X,\bar{Y}) = -\eta(Y)\bar{Y} \tag{5.6}$$

By virtue of the equations (5.5) and (5.6), the result (5.1) (v) follows.

Theorem 5.2. In a Hsu-unified structure manifold Mn with a Recurrent metric connection B, we have

$$\left.\begin{array}{l} (i)\ P(\bar{X},\bar{Y},Z) = -\eta(\bar{X})F(Y,Z) = P(\bar{X},Y,\bar{Z}) \\ (ii)\ P(\bar{X},\bar{Y},Z) = -\eta(\bar{X})F(Z,Y) = P(\bar{X},\bar{Z},Y) \\ (iii)\ P(\bar{X},\bar{Y},\bar{Z}) = a^r P(\bar{X},Y,Z) \end{array}\right\} \tag{5.7}$$

Proof: Barring X and Y in equation (3.8) and using the equation (2.3) , we get

$$P(\bar{X},\bar{Y},Z) = -\eta(\bar{X})F(Y,Z) \tag{5.8}$$

Again barring X and Z in equation (3.8) and using the equation (2.3) we obtain

$$P(\bar{X},Y,\bar{Z}) = -\eta(\bar{X})F(Y,Z) \tag{5.9}$$

Equation (5.7)(i) follows from equation(5.8) and (5.9) and equation(5.7)(ii) follows from equations (2.6) , (5.8) and (5.9).

Barring Z in equation (5.8), we have

$$P(\bar{X},\bar{Y},\bar{Z}) = -\eta(\bar{X})F(Y,\bar{Z}) \tag{5.10}$$

Barring X in equation (3.8), we have

$$P(\bar{X},Y,Z) = -\eta(\bar{X})g(Y,Z) \tag{5.11}$$

In view of the equations (5.10),(5.11) and (2.5), equation (5.7) (iii) follows easily.

Theorem 5.3. A Hsu-unified structure manifold Mn with a Recurrent metric connection B satisfies the following relation

$$\left.\begin{array}{l} (i)\ (B_X\phi)Y = (D_X\phi)Y \\ (ii)\ (B_X\eta)Y = (D_X\eta)Y \\ (iii)\ (B_{\bar{X}}\phi)\bar{Y} = (B_X\phi)Y \ \ iff \ \ (D_{\bar{X}}\phi)\bar{Y} = (D_X\phi)Y \end{array}\right\} \tag{5.12}$$

Proof : Replacing Y by in equation (3.1), we have

$$B_X\bar{Y} = D_X\bar{Y} - \eta(X)\bar{Y} \tag{5.13}$$

The above equation can also be modified as

$$(B_X\phi)Y = (D_X\phi)\bar{Y} - \eta(X)\bar{Y} - \overline{B_XY} \tag{5.14}$$

Operating φ on both sides of the equation (3.1) yields

$\overline{B_X Y} = \overline{D_X Y} - \eta(X)\bar{Y}$ (5.15)

From the equations (5.14) and (5.15), we get

$(B_X\phi)Y = (D_X\phi)\bar{Y} - \overline{D_X Y}$ (5.16)

We have

$D_X\bar{Y} = \overline{D_X Y} + (D_X\phi)Y.$ (5.17)

From the equations (5.16) and (5.17) ,we obtain (5.12) (i)

$(B_X\phi)Y = (D_X\phi)Y$ (5.18)

Barring X and Y in equation (5.18)

$(B_{\bar{X}}\phi)\bar{Y} = (D_{\bar{X}}\phi)\bar{Y}$ (5.19)

Subtracting equation (5.18) from (5.19), we have

$(B_{\bar{X}}\phi)\bar{Y} - (B_X\phi)Y = (D_{\bar{X}}\phi)\bar{Y} - (D_X\phi)Y$ (5.20)

clearly, equation (5.20) proves equation (5.12)(iii).

From equation (5.14),we have

$(B_X\eta)Y = \eta(B_X Y) - B_X(\eta(Y))$ (5.21)

From equations (3.1) and (5.21)

$(B_X\eta)Y = \eta(D_X Y) - D_X(\eta(Y)) = (D_X\eta)Y$

Theorem 5.4. If a Hsu-unified structure manifold Mn admits semi-symmetric metric connection B, then the Nijenhuis tensors of the Riemannian connection D and B coincide.

Proof : The Nijenhuis tensor with respect to ∅ is a vector valued bilinear function, defined as[3].

$\tilde{N}(X,Y) = [\bar{X},\bar{Y}] - [\overline{\bar{X},Y}] - [\overline{X,\bar{Y}}] + [\overline{\overline{X,Y}}]$ (5.22)

In view of the equation (2.1), the above expression can be written in the form

$\tilde{N}(X,Y) = [\bar{X},\bar{Y}] - [\overline{\bar{X},Y}] - [\overline{X,\bar{Y}}] + a^r[X,Y]$ (5.23)

The Nijenhuis tensor with respect to the Riemannian connection D is given

As $N(X,Y) = (D_{\bar{X}}\emptyset)Y - (D_{\bar{Y}}\emptyset)X - \overline{(D_X\emptyset)Y} + \overline{(D_Y\emptyset)X}$ (5.24)

By virtue of the equations (5.18) and (5.19), (5.24) becomes

$N(X,Y) = (B_{\bar{X}}\emptyset)Y - (B_{\bar{Y}}\emptyset)X - \overline{(B_X\emptyset)Y} + \overline{(B_Y\emptyset)X}$ (5.25)

Where $\tilde{N}(X,Y)$ denotes the Nijenhuis tensor with respect to the quarter-symmetric metric connection B. From equations (5.22) and (5.25) we have the statement of the theorem.

Theorem 5.5. A Hsu-unified structure manifold Mn admits semi-symmetric non- metric recurrent connection B, such that BXF = 0, then F is killing if and only if

$\eta(X)F(Y,Z) + \eta(Y)F(X,Z) = 0$

Proof: We have

$B_X(F(Y,Z)) = (B_X F)(Y,Z) + F(B_X Y,Z) + F(Y,B_X Z)$ (5.26)

$(B_X F)(Y,Z) = B_X(F(Y,Z)) - F(B_X Y,Z) - F(Y,B_X Z)$ (5.27)

using equation (3.1) in the equation (5.26), we have

$(B_X F)(Y,Z) = (D_X F)(Y,Z) - \eta(X)F(Y,Z)$ (5.28)

$(D_X F)(Y,Z) = (B_X F)(Y,Z) + \eta(X)F(Y,Z)$ (5.28)

Similarly,

$(D_Y F)(X,Z) = (B_Y F)(X,Z) + \eta(Y)F(X,Z)$ (5.30)

Adding equations (5.28) and (5.29), we get

$(D_X F)(Y,Z) + (D_Y F)(X,Z) = \eta(X)F(Y,Z) + \eta(Y)F(X,Z)$

$\widetilde{N}(X,Y,Z) = (B_{\bar{X}} F)(Y,Z) - (B_{\bar{Y}} F)(X,Z) + (B_X F)(Y,\bar{Z}) - (B_Y F)(X,\bar{Z})$

Which proves the statement.

6. HSU-KAHLER MANIFOLD WITH THE SEMI SYMMETRIC NON-METRIC CONNECTION

As discussed earlier that if the Hsu-unified structure manifold M_n satisfies the condition (2.7), then M_n is called a Hsu-Kahler manifold. In this section, we have following theorems.

Theorem 6.1. If M_n be a Hsu-Kahler manifold admitting a semi-symmetric metric connection B then in M_n

$(B_X \emptyset)Y = 0$ (6.1)

Proof. In a Hsu-Kahler manifold M_n , the equations (5.14) and (2.7) follows

$(B_X \emptyset)Y = \overline{D_X Y} - \eta(X)\bar{Y} - \overline{B_X Y}$ (6.2)

From the equations (6.2) and (5.15), we have the statement of the theorem

$(B_X \emptyset)Y = \overline{D_X Y} - \eta(X)\bar{Y} - \overline{B_X Y}$.

Theorem 6.2. A Hsu-Kahler manifold Mn with a semi symmetric metric connection B satisfies the following relation

$dF(X,Y,Z) = 2[\eta(Xg(Y,Z) + \eta(Y)g(Z,X) + \eta(Z)g(X,Y)]$ (6.3)

Proof: By definition, we know that

$dF(X,Y,Z) = (B_X F)(Y,Z) + (B_Y F)(X,Z) + (B_Z F)(X,Y)$ (6.4)

From the equation (2.3), we have

$F(Y,Z) = g(\bar{Y},Z)$ (6.5)

Differentiating the equation (6.5) covariantly with respect to X and using the expressions of the equations (2.3), (2.8), (3.1) and (3.3), we obtain,

$(B_X F)(Y,Z) = 2\eta(X)g(\bar{Y},Z).$ (6.6)

Proceeding in the same way , we have

$(B_Y F)(Z,X) = 2\eta(X)g(\bar{Z},X)$ (6.7)

$(B_Z F)(X,Y) = 2\eta(Z)g(\bar{X},Y)$ (6.8)

In consequence with the equations (6.4), (6.6), (6.7) and (6.8), the required result follows.

Theorem 6.3. Curvature tensor of semi symmetric non metric recurrent connection B in a Hsu-Kahler manifold is given by

$\tilde{R}(X,Y,Z) = R(X,Y,Z) - \theta_Y^X + \theta_X^Y$, (6.9)

Where R and $\tilde{R}$ be the curvature tensor with respect to the connection B and D respectively and $\theta_Y^X = [(D_X\eta)Y]Z.$

Proof. The curvature tensor with respect to the semi symmetric non metric connection B is defined as

$$\tilde{R}(X,Y,Z) = B_X B_Y Z - B_Y B_X Z - B_{[X,Y]} Z. \quad (6.10)$$

Using equation (3.1) in the equation (6.10), we obtain

$$\tilde{R}(X,Y,Z) = B_X[D_Y Z - \eta(Y)Z] - B_Y[D_X Z - \eta(X)Z] - [D_{[X,Y]} Z - \eta([X,Y])Z]$$

We also know that

$$[X,Y] = D_X Y - D_Y X.$$

From equations (6.11) and (6.12),we get

$$\tilde{R}(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z - [(D_X\eta)Y]Z - [(D_Y\eta)X]Z = R(X,Y,Z) - \theta_Y^X + \theta_X^Y \quad (6.13)$$

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