

# On a Hsu-unified Structure Manifold with a Recurrent Metric Connection

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## ABSTRACT

In the present paper, we have defined a Hsu-unified structure manifold and a Hsu-Kahler manifold and studied some properties of the recurrent metric connection in a Hsu-unified structure Manifold. We have also studied the curvature tensor of Hsu-Kahler manifold with respect to the recurrent metric connection.

**Keywords:** Semi-symmetric recurrent metric connection, Generalized co-symplectic manifold, Generalized quasi-Sasakian manifold.

## 1. INTRODUCTION

Recurrent metric connection was introduced and studied by Liang, Y<sup>2</sup> in 1994. Several properties of semi-symmetric metric and non-metric connections on a differentiable manifold have been studied by Yano and Imai<sup>5</sup>, Agashe and Chafle<sup>1</sup> and many others. In the present paper, we have studied some properties of the semi-symmetric non metric connection on a Hsu-unified structure manifold. It has been shown that the Nijenhuis tensor with respect to semi-symmetric non-metric connection and with respect to Riemannian connection coincide in the Hsu-unified structure manifold but in the Hsu-Kahler manifold Nijenhuis tensor with respect to semi-symmetric metric connection vanishes identically i.e. a Hsu-Kahler manifold is integrable.

## 2. PRELIMINARIES

An even dimensional differentiable manifold  $M_n(n = 2m)$ , there exists a vector valued linear function  $\phi$  of differentiability class  $C^\infty$  satisfying

$$\phi^2 X = a^r X \tag{2.1}$$

for arbitrary vector field  $X$ . Also there exists a Riemannian metric  $g$ , such that

$$g(\bar{X}, \bar{Y}) = a^r g(X, Y) \tag{2.2}$$

where  $\bar{X} = \phi X$ ,  $0 \leq r \leq n$  and  $a$  is real or complex number.  
Then  $M_n$  is said to be a Hsu-unified structure manifold.

Let us define a 2-form  $F$  in  $M_n$

$$F(X, Y) = g(\bar{X}, \bar{Y}) = g(X, \bar{Y}) \tag{2.3}$$

Then it is clear that the 2-form  $F$  satisfies

$$F(\bar{X}, \bar{Y}) = a^r g(X, Y) \tag{2.4}$$

From equation (2.3), we have the following properties

$$F(\bar{X}, Y) = a^r g(X, Y) \tag{2.5}$$

$$F(X, Y) = F(Y, X) \tag{2.6}$$

Equation (2.6) shows that the 2-form  $F$  is symmetric in  $M_n$ .  
If the Hsu-unified structure manifold  $M_n$  satisfies a condition

$$D_X \bar{Y} = \bar{D}_X \bar{Y} \Leftrightarrow \overline{D_X Y} = a^r (D_X Y) \tag{2.8}$$

### 3. SEMI SYMMETRIC RECURRENT METRIC CONNECTION

A linear connection  $B$  defined as<sup>4</sup>

$$B_X Y = D_X Y - \eta(X, Y) \tag{3.1}$$

For arbitrary vector fields  $X$  and  $Y$  is said to be a semi –symmetric recurrent metric connection if the torsion tensor  $S$  of the connection  $B$  and the metric tensor  $g$  are given by

$$S(X, Y) = \eta(Y)X - \eta(X)Y \tag{3.2}$$

And

$$(B_X g)(Y, Z) = 2\eta(X)g(Y, Z) \tag{3.3}$$

where  $\eta$  is one form associated with the vector field  $\xi$  such that

$$\eta(X) = g(X, \xi) \tag{3.4}$$

Putting equation (3.1) as

$$B_X Y = D_X Y + P(X, Y) \tag{3.5}$$

Then

$$P(X, Y) = -\eta(X)Y \tag{3.6}$$

Let us define

$$P(X, Y, Z) = -g(P(X, Y), Z) \tag{3.7}$$

then by virtue of the equation (3.6), equation (3.7) becomes

$$P(X, Y, Z) = -\eta(X)g(Y, Z) \tag{3.8}$$

#### 4. EXISTENCE OF SEMI SYMMETRIC RECURRENT METRIC CONNECTION

A linear connection B defined as<sup>4</sup>

$$B_X Y = D_X Y + P(X, Y) \tag{4.1}$$

where P is a tensor of type (1,2) for arbitrary vector fields X and Y is said to be a semi-symmetric recurrent metric connection if the torsion tensor S of the connection B and the metric tensor g are given by

$$S(X, Y) = P(X, Y) - P(Y, X) \tag{4.2}$$

And

$$G(X, Y, Z) = (B_X g)(Y, Z) \tag{4.3}$$

$$g(P(X, Y), Z) + g(P(X, Z), Y) = -G(X, Y, Z) \tag{4.4}$$

From equations (4.1), (4.2) and (4.4), we have

$$\begin{aligned} &g(S(X, Y), Z) + g(S(Z, X), Y) + g(S(Z, Y), X) \\ &= g(P(X, Y), Z) - g(P(Y, X), Z) + g(P(Z, X), Y) - g(P(X, Z), Y) + g(P(Z, Y), X) \\ &\quad - g(P(Y, Z), X) \\ &= 2g(P(X, Y), Z) + 2\eta(X)g(Y, Z) + 2\eta(Y)g(X, Z) - 2\eta(Z)g(X, Y) \end{aligned}$$

$$P(X, Y) = \frac{1}{2}\{S(X, Y) + \check{S}(X, Y) + \check{S}(Y, X)\} - \eta(X)Y - \eta(Y)X = g(X, Y) \tag{4.5}$$

where

$$g(\check{S}(X, Y), Z) = g(S(Z, X)Y). \tag{4.6}$$

From equations (4.6) and (3.4), we get

$$\check{S}(X, Y) = \eta(X)Y - g(X, Y)\xi \tag{4.7}$$

Then in view of equations (4.5) and (4.7), we get

$$P(X, Y) = -\eta(X)Y \tag{4.8}$$

Which implies

$$B_X Y = D_X Y - \eta(X, Y) \tag{4.9}$$

#### 5. HSU-UNIFIED STRUCTURE MANIFOLD EQUIPPED WITH SEMI SYMMETRIC RECURRENT METRIC CONNECTION

In this section, we have the following theorems:

**Theorem 5.1.** For a Hsu-unified structure manifold  $M_n$  equipped with a semi symmetric recurrent metric connection, the following results hold good

$$\left. \begin{aligned} (i) P(\bar{X}, \bar{Y}) &= a^r P(\bar{X}, Y) \\ (ii) P(\bar{X}, \bar{Y}) &= a^r P(X, \bar{Y}) \\ (iii) P(\bar{X}, Y) &= P(X, \bar{Y}) = a^r P(X, Y) \\ (iv) P(X, \bar{Y}) &= P(Y, \bar{X}) \text{ iff } \eta(X)Y = \eta(Y)X \\ (v) P(X, \bar{Y}) &= P(Y, \bar{X}) \text{ iff } \eta(X)\bar{Y} = \eta(\bar{X})Y \end{aligned} \right\} \quad (5.1)$$

**Proof :** From the equations (3.6) and (2.1), we have

$$\left. \begin{aligned} (i) P(\bar{X}, \bar{Y})Y &= -a^r \eta(\bar{X})Y \\ (ii) P(\bar{X}, \bar{Y}) &= -a^r \eta(X)\bar{Y} \\ (iii) P(\bar{X}, Y) &= P(X, \bar{Y}) = -a^r \eta(X)Y \end{aligned} \right\} \quad (5.2)$$

Clearly equations (5.1) (i) , (ii)and (iii) follows from equations (5.2) (i), (5.2) (ii) and (5.2) (iii). Interchanging X and Y in equation (3.6), we get

$$P(Y, X) = -\eta(Y)X \quad (5.3)$$

Barring X in equation (5.3) and using equation (2.1), we have

$$P(Y, \bar{X}) = -a^r \eta(Y)X \quad (5.4)$$

Now the result of (5.1) (iv) follows from equations (5.2) (iii) and (5.4). Again barring X in equation (5.3),we have

$$P(Y, \bar{X}) = -\eta(Y)\bar{X} \quad (5.5)$$

From the equations (3.6) and (2.1), we have

$$P(X, \bar{Y}) = -\eta(Y)\bar{Y} \quad (5.6)$$

By virtue of the equations (5.5) and (5.6), the result (5.1) (v) follows.

**Theorem 5.2.** In a Hsu-unified structure manifold Mn with a Recurrent metric connection B, we have

$$\left. \begin{aligned} (i) P(\bar{X}, \bar{Y}, Z) &= -\eta(\bar{X})F(Y, Z) = P(\bar{X}, Y, \bar{Z}) \\ (ii) P(\bar{X}, \bar{Y}, Z) &= -\eta(\bar{X})F(Z, Y) = P(\bar{X}, \bar{Z}, Y) \\ (iii) P(\bar{X}, \bar{Y}, \bar{Z}) &= a^r P(\bar{X}, Y, Z) \end{aligned} \right\} \quad (5.7)$$

**Proof:** Barring X and Y in equation (3.8) and using the equation (2.3) , we get

$$P(\bar{X}, \bar{Y}, Z) = -\eta(\bar{X})F(Y, Z) \quad (5.8)$$

Again barring X and Z in equation (3.8) and using the equation (2.3) we obtain

$$P(\bar{X}, Y, \bar{Z}) = -\eta(\bar{X})F(Y, Z) \quad (5.9)$$

Equation (5.7)(i) follows from equation(5.8) and (5.9) and equation(5.7)(ii) follows from equations (2.6) , (5.8) and (5.9).

Barring Z in equation (5.8), we have

$$P(\bar{X}, \bar{Y}, \bar{Z}) = -\eta(\bar{X})F(Y, \bar{Z}) \quad (5.10)$$

Barring X in equation (3.8), we have

$$P(\bar{X}, Y, Z) = -\eta(\bar{X})g(Y, Z) \tag{5.11}$$

In view of the equations (5.10),(5.11) and (2.5), equation (5.7) (iii) follows easily.

**Theorem 5.3.** A Hsu-unified structure manifold Mn with a Recurrent metric connection B satisfies the following relation

$$\left. \begin{aligned} (i) (B_X\phi)Y &= (D_X\phi)Y \\ (ii) (B_X\eta)Y &= (D_X\eta)Y \\ (iii) (B_{\bar{X}}\phi)\bar{Y} &= (B_X\phi)Y \text{ iff } (D_{\bar{X}}\phi)\bar{Y} = (D_X\phi)Y \end{aligned} \right\} \tag{5.12}$$

**Proof :** Replacing Y by in equation (3.1), we have

$$B_X\bar{Y} = D_X\bar{Y} - \eta(X)\bar{Y} \tag{5.13}$$

The above equation can also be modified as

$$(B_X\phi)Y = (D_X\phi)\bar{Y} - \eta(X)\bar{Y} - \bar{B}_X\bar{Y} \tag{5.14}$$

Operating  $\phi$  on both sides of the equation (3.1) yields

$$\bar{B}_X\bar{Y} = \bar{D}_X\bar{Y} - \eta(X)\bar{Y} \tag{5.15}$$

From the equations (5.14) and (5.15), we get

$$(B_X\phi)Y = (D_X\phi)\bar{Y} - \bar{D}_X\bar{Y} \tag{5.16}$$

We have

$$D_X\bar{Y} = \bar{D}_X\bar{Y} + (D_X\phi)Y. \tag{5.17}$$

From the equations (5.16) and (5.17), we obtain (5.12) (i)

$$(B_X\phi)Y = (D_X\phi)Y \tag{5.18}$$

Barring X and Y in equation (5.18)

$$(B_{\bar{X}}\phi)\bar{Y} = (D_{\bar{X}}\phi)\bar{Y} \tag{5.19}$$

Subtracting equation (5.18) from (5.19), we have

$$(B_{\bar{X}}\phi)\bar{Y} - (B_X\phi)Y = (D_{\bar{X}}\phi)\bar{Y} - (D_X\phi)Y \tag{5.20}$$

clearly, equation (5.20) proves equation (5.12)(iii).

From equation (5.14),we have

$$(B_X\eta)Y = \eta(B_XY) - B_X(\eta(Y)) \tag{5.21}$$

From equations (3.1) and (5.21)

$$\begin{aligned} (B_X\eta)Y &= \eta(D_XY) - D_X(\eta(Y)) \\ &= (D_X\eta)Y \end{aligned}$$

**Theorem 5.4.** If a Hsu-unified structure manifold Mn admits semi-symmetric metric connection B , then the Nijenhuis tensors of the Riemannian connection D and B coincide.

**Proof :** The Nijenhuis tensor with respect to  $\phi$  is a vector valued bilinear function, defined as<sup>3</sup>.

$$\tilde{N}(X, Y) = [\bar{X}, \bar{Y}] - [\bar{X}, Y] - [\bar{X}, \bar{Y}] + [\bar{X}, \bar{Y}] \tag{5.22}$$

In view of the equation (2.1), the above expression can be written in the form

$$\tilde{N}(X, Y) = [\bar{X}, \bar{Y}] - [\bar{X}, Y] - [\bar{X}, \bar{Y}] + a^r [X, Y] \tag{5.23}$$

The Nijenhuis tensor with respect to the Riemannian connection D is given

$$\text{As } N(X, Y) = (D_{\bar{X}}\phi)Y - (D_{\bar{Y}}\phi)X - \overline{(D_X\phi)Y} + \overline{(D_Y\phi)X} \tag{5.24}$$

By virtue of the equations (5.18) and (5.19), (5.24) becomes

$$N(X, Y) = (B_{\bar{X}}\phi)Y - (B_{\bar{Y}}\phi)X - \overline{(B_{\bar{X}}\phi)Y} + \overline{(B_{\bar{Y}}\phi)X} \quad (5.25)$$

Where  $\tilde{N}(X, Y)$  denotes the Nijenhuis tensor with respect to the quarter-symmetric metric connection B. From equations (5.22) and (5.25) we have the statement of the theorem.

**Theorem 5.5.** A Hsu-unified structure manifold Mn admits semi-symmetric non- metric recurrent connection B, such that  $BXF = 0$ , then F is killing if and only if  $\eta(X)F(Y, Z) + \eta(Y)F(X, Z) = 0$

**Proof.** We have

$$B_X(F(Y, Z)) = (B_X F)(Y, Z) + F(B_X Y, Z) + F(Y, B_X Z) \quad (5.26)$$

$$(B_X F)(Y, Z) = B_X(F(Y, Z)) - F(B_X Y, Z) - F(Y, B_X Z) \quad (5.27)$$

using equation (3.1) in the equation (5.26), we have

$$(B_X F)(Y, Z) = (D_X F)(Y, Z) - \eta(X)F(Y, Z) \quad (5.28)$$

$$(D_X F)(Y, Z) = (B_X F)(Y, Z) + \eta(X)F(Y, Z) \quad (5.28)$$

Similarly,

$$(D_Y F)(X, Z) = (B_Y F)(X, Z) + \eta(Y)F(X, Z) \quad (5.30)$$

Adding equations (5.28) and (5.29), we get

$$(D_X F)(Y, Z) + (D_Y F)(X, Z) = \eta(X)F(Y, Z) + \eta(Y)F(X, Z)$$

$$\tilde{N}(X, Y, Z) = (B_{\bar{X}} F)(Y, Z) - (B_{\bar{Y}} F)(X, Z) + (B_X F)(Y, \bar{Z}) - (B_Y F)(X, \bar{Z})$$

Which proves the statement.

## 6. HSU-KAHLER MANIFOLD WITH THE SEMI SYMMETRIC NON-METRIC CONNECTION

As discussed earlier that if the Hsu-unified structure manifold  $M_n$  satisfies the condition (2.7), then  $M_n$  is called a Hsu-Kahler manifold. In this section, we have following theorems.

**Theorem 6.1.** If  $M_n$  be a Hsu-Kahler manifold admitting a semi-symmetric metric connection B then in  $M_n$

$$(B_X \phi)Y = 0 \quad (6.1)$$

Proof. In a Hsu-Kahler manifold  $M_n$ , the equations (5.14) and (2.7) follows

$$(B_X \phi)Y = \overline{D_X Y} - \eta(X)\bar{Y} - \overline{B_X Y} \quad (6.2)$$

From the equations (6.2) and (5.15), we have the statement of the theorem

$$(B_X \phi)Y = \overline{D_X Y} - \eta(X)\bar{Y} - \overline{B_X Y} .$$

**Theorem 6.2.** A Hsu-Kahler manifold Mn with a semi symmetric metric connection B satisfies the following relation

$$dF(X, Y, Z) = 2[\eta(X)g(Y, Z) + \eta(Y)g(Z, X) + \eta(Z)g(X, Y)] \quad (6.3)$$

**Proof.** By definition, we know that

$$dF(X, Y, Z) = (B_X F)(Y, Z) + (B_Y F)(X, Z) + (B_Z F)(X, Y) \quad (6.4)$$

From the equation (2.3), we have

$$F(Y, Z) = g(\bar{Y}, Z) \quad (6.5)$$

Differentiating the equation (6.5) covariantly with respect to X and using the expressions of the equations (2.3), (2.8), (3.1) and (3.3), we obtain,

$$(B_X F)(Y, Z) = 2\eta(X)g(\bar{Y}, Z). \quad (6.6)$$

Proceeding in the same way, we have

$$(B_Y F)(Z, X) = 2\eta(X)g(\bar{Z}, X) \quad (6.7)$$

$$(B_Z F)(X, Y) = 2\eta(Z)g(\bar{X}, Y) \quad (6.8)$$

In consequence with the equations (6.4), (6.6), (6.7) and (6.8), the required result follows.

**Theorem 6.3.** Curvature tensor of semi symmetric non metric recurrent connection B in a Hsu-Kahler manifold is given by

$$\tilde{R}(X, Y, Z) = R(X, Y, Z) - \theta_Y^X + \theta_X^Y, \quad (6.9)$$

Where R and  $\tilde{R}$  be the curvature tensor with respect to the connection B and D respectively and

$$\theta_Y^X = [(D_X \eta)Y]Z.$$

**Proof.** The curvature tensor with respect to the semi symmetric non metric connection B is defined as

$$\tilde{R}(X, Y, Z) = B_X B_Y Z - B_Y B_X Z - B_{[X, Y]}Z. \quad (6.10)$$

Using equation (3.1) in the equation (6.10), we obtain

$$\tilde{R}(X, Y, Z) = B_X [D_Y Z - \eta(Y)Z] - B_Y [D_X Z - \eta(X)Z] - [D_{[X, Y]}Z - \eta([X, Y])Z]$$

We also know that

$$[X, Y] = D_X Y - D_Y X.$$

From equations (6.11) and (6.12), we get

$$\tilde{R}(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z - [(D_X \eta)Y]Z - [(D_Y \eta)X]Z = R(X, Y, Z) - \theta_Y^X + \theta_X^Y \quad (6.13)$$

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