On the Hyper-Wiener Index of Graph Amalgamation

Shigehalli V. S\textsuperscript{1}, D. N. Misale\textsuperscript{2} and Shanmukh Kuchabal\textsuperscript{3}

\textsuperscript{1}Senior Associate Professor  
Head, Dept. of Physics  
Bhaurao Kakatkar College, Belgaum, INDIA.  
\textsuperscript{2}Professor  
Chairman and Head of the Department  
\textsuperscript{3}Dept. of Mathematics,  
Rani Channamma University,  
Vidyasangama, Belagavi, INDIA.

(Received on: July 17, 2014)

ABSTRACT

Let $G$ be the graph. The Wiener Index $W(G)$ is the sum of all distances between vertices of $G$, whereas the Hyper-Wiener index $WW(G)$ is defined as $WW(G) = \frac{1}{2} W(G) + \frac{1}{2} \sum d^2(u,v)$. In this paper we prove results on Hyper-Wiener Index of Amalgamation of Complete Graph $K_p$ with common vertex and Amalgamation of cyclic graph $C_4$ with common edge.

Keywords: Hyper-Wiener Index, Amalgamation of Complete Graph $K_p$, Amalgamation of cyclic graph $C_4$.

1. INTRODUCTION

We have three methods for calculation of the Hyper-Wiener Index of molecular graphs.

(i) Distance Formula:

$$WW(G) = \frac{1}{2}(\sum d(u, v) + \sum d^2(u,v))$$

(ii) Cut Method:

(iii) The Method of Hosaya Polynomials:

1.1 Cut Method

The cut method is based on the results from Klavzar, Gutman and Mohar and the calculation of the hyper-Wiener index works for all partial cubes. A graph is a partial cube if it is isomorphic to an isometric subgraph of a hypercube. Let $G$ be a benzenoid graph on vertices. Then an elementary cut $C$ divides $G$ into two components, say $G_1(C)$ and $G_2(C)$. 
Let $n_1(C)$ and $n_2(C)$ be the number of vertices of $G_1(C)$ and $G_2(C)$, respectively. Then Wiener index of $G$ can be calculated as:

\[ W(G) = \sum_i n_1(C_i) n_2(C_i) \quad (1) \]

where the summation goes over all elementary cuts of $G$.

The hyper-Wiener index of a benzenoid graph $G$ can be written as:

\[ WW(G) = W(G) + WW^*(G) \quad (2) \]

Where $WW^*(G)$ consists of a summation over all pairs of elementary cuts. Let $C_1$ and $C_2$ be two elementary cuts of a benzenoid graph $G$. There are two different cases, as shown in figure (a). With $a$, $b$, $c$ and $d$ we denote the number of vertices in the corresponding parts of $G$. Then the contribution of the pair $C_1$, $C_2$ to $WW^*(G)$ is $ab+cd$ in the first case and $ab$ in the second one.

**1.2 Definitions**

**Graph Amalgamation:** Let $H$ be a graph then graph Amalgamation $G$ obtained by joining two or more copies of the same graph $H$ either with common vertex or with common edge.

In this paper we present two main Theorems. The first Theorem proved based on distance formula and second theorem by cut method.

**2. MAIN RESULT**

**Theorem 2.1** Let $K_p$ be the Complete graph containing $p$ vertices then graph Amalgamation $G$ obtained by joining same graph $K_p$ to each vertex of $K_p$ with common vertex then its Hyper-Wiener index given by

\[ WW(G) = \frac{1}{2} \{ 6n^2 - 10n |K_p| + 4 |K_p|^2 |K_p| (n+1) + 2n \} \]

Where $n$ - number of vertices in graph $G$ $K_p$ - Complete graph containing $P$ vertices $|K_p|$ - Cardinality of Complete graph

**Proof:** To find Hyper-Wiener Index of graph we need to find following two parts
To find $W(G)$:

\[ W(G) = \frac{1}{2} \sum d(u,v) \]

\[ \frac{1}{2} \{ [(1+1+\ldots+1)+(2+2+\ldots+2)+(3+3+\ldots+3)+\ldots+(1+1+\ldots+1)+(2+2+\ldots+2)+(3+3+\ldots+3)] 
\]

\[ |K_p| -1 \text{ times} \quad [n-(2|K_p|-1)] \text{ times} \quad |K_p| -1 \text{ times} \quad [n-(2|K_p|-1)] \text{ times} \]

\[ + [(1+1+\ldots+1)+(2+2+\ldots+2)+\ldots\ldots\ldots\ldots+(1+1+\ldots+1)+(2+2+\ldots+2)] \]
\[ \begin{align*} &2 \mid K_p \mid -2 \text{ times} \quad [n-(2 \mid K_p \mid -1)] \text{ times} \quad 2 \mid K_p \mid -2 \text{ times} \quad [n-(2 \mid K_p \mid -1)] \text{ times} \\
&= \frac{1}{2} \left[ \left( \mid K_p \mid -1 \right)^3 + [n-(2 \mid K_p \mid -1)]^3 + \ldots + \left( \mid K_p \mid -1 \right)^3 + [n-(2 \mid K_p \mid -1)]^3 \right] + \\
&\quad \left( n \mid K_p \mid \right) \text{ times} \\
&\frac{1}{2} \left[ [2 \mid K_p \mid -2] + [n-(2 \mid K_p \mid -1)] \right] \quad \left( n \mid K_p \mid \right) \text{ times} \\
&= \frac{1}{2} \left\{ (n \mid K_p \mid) \left( \mid K_p \mid -1 \right)^3 + (n \mid K_p \mid) \left[ n-(2 \mid K_p \mid -1) \right]^3 + \frac{1}{2} \left\{ [2 \mid K_p \mid -2] + [n-(2 \mid K_p \mid -1)] \right\} \right\} \\
&\frac{1}{2} \left( n \mid K_p \mid \right) \text{ times} \\
&W(G) = \frac{1}{2} \{ (n-3 \mid K_n \mid) + \mid K_n \mid (\mid K_n \mid -n) \} \quad \text{---(i)} \\
\end{align*} \]

To find \( WW^*(G) \):
\[ \begin{align*} WW^* (G) &= \frac{1}{2} \sum d^2(u,v) \\
&= \frac{1}{2} \left\{ [(1+1+\ldots+1)+(3+3+\ldots+3)+\ldots+(1+1+\ldots+1)+(3+3+\ldots+3)] + \\
&\quad \left( \mid K_p \mid -1 \right) \text{ times} \quad \left[ n-(2 \mid K_p \mid -1) \right] \text{ times} \quad \left( \mid K_p \mid -1 \right) \text{ times} \quad \left[ n-(2 \mid K_p \mid -1) \right] \text{ times} \\
&\quad \left[ (1+1+\ldots+1)+\ldots+(1+1+\ldots+1) \right] \left( \mid K_p \mid -1 \right) \text{ times} \quad \left[ n-(2 \mid K_p \mid -1) \right] \text{ times} \\
&= \frac{1}{2} \left\{ \left[ \mid K_p \mid -1 \right] + [n-(2 \mid K_p \mid -1)]^3 + \ldots + \left[ \mid K_p \mid -1 \right] + [n-(2 \mid K_p \mid -1)]^3 + \\
&\quad \left[ n-(2 \mid K_p \mid -1) \right] + [n-(2 \mid K_p \mid -1)] + \ldots + [n-(2 \mid K_p \mid -1)] \right\} \\
&\frac{1}{2} \left( \mid K_p \mid \right) \text{ times} \\
&WW^* (G) = \frac{1}{2} \left\{ \left[ \mid K_p \mid -1 \right](n \mid K_p \mid) + [n-(2 \mid K_p \mid -1)]^3 (n \mid K_p \mid) + \mid K_p \mid \left[ n-(2 \mid K_p \mid -1) \right] \right\} \quad \text{---(ii)} \\
\end{align*} \]

Since \( WW(G) = W(G) + WW^*(G) \)
Combining (i) and (ii) gives

\[ \text{Journal of Computer and Mathematical Sciences Vol. 5, Issue 4, 31 August, 2014 Pages (332-411)} \]
WW(G) = \frac{1}{2} \{6n^2 - 10n \mid K_p \} + 4 \mid K_p \mid (n+1) + 2n \}

Illustrations:

Lemma 2.2: In the following theorem there are \( \frac{n}{2} \) number of elementary cuts present in W(G).

Lemma 2.3: In the following theorem there are \( \frac{n}{4}(\frac{n}{2} - 1) \) number of elementary cuts present in WW*(G).

Theorem 2.4: Let \( H_1, H_2, \ldots, H_m \) be the cyclic graph containing four vertices then graph Amalgamation G obtained from joining \( H_1, H_2, \ldots, H_m \) with common edge then its Hyper-Wiener Index given by

\[
WW(G) = \sum_{i=2}^{n-1} \frac{n}{2} \binom{i}{2} l_j + \frac{1}{4} l_j^2 + 2\sum_{k=1}^{n-1} \sum_{l=1}^{n-1} kl + 2 \sum_{i=2}^{n-4} l_1 + 4 \sum_{l_2=2}^{n-6} l_2 + \ldots + (n-4) \sum_{l_{i-2}=2}^{n-4} l_i
\]

Where i, j, l_1, l_2, \ldots, l_i are 2, 4, 6, 8, \ldots, \ldots, and k, l are 1, 2, 3, \ldots, \ldots

Proof: we write the proof using cut method, to find Hyper-Wiener Index of graph we need to find following two parts

(i) To find W(G):

\[
W(G) = \sum_{i=2}^{n-2} \binom{i}{2} l_j + \frac{1}{4} l_j^2 + n\binom{C_1}{n} + n\binom{C_2}{n} + \ldots + n\binom{C_l}{n} = 2(n-2) + 4(n-4) + 6(n-6) + \ldots + \binom{C_{i-1}}{n} + \binom{C_i}{n}
\]

(ii) To find WW*(G):

Since there are \( C_1 C_2, C_1 C_3, \ldots, C_{l_i} \) distinct pair cuts in WW*(G)

\[
WW*(G) = \binom{C_1 C_2}{n} + \binom{C_1 C_3}{n} + \ldots + \binom{C_{l_i} C_i}{n}
\]

= \left( \frac{n}{2} - 1 \right) + \left( \frac{n}{2} - 1 \right) + 2\left( \frac{n}{2} - 2 \right) + 2\left( \frac{n}{2} - 2 \right) + \ldots +
\[
\begin{align*}
\frac{n}{2} - 1 \quad + \quad \frac{n}{2} - 1 \quad + \quad 2[2+4+6+\ldots+(n-4)] \\
+4[2+4+6+\ldots+(n-6)] + \ldots + (n-4) \quad + \quad (n-3) \quad + \quad \ldots + (n-4)
\end{align*}
\]

Since \( WW(G) = W(G) + WW^*(G) \)

Combining (i) and (ii) gives

\[
WW(G) = \sum_{i=2}^{n-2} \frac{n-2}{4} i j + \frac{1}{4} n^2 + \frac{2}{2} \sum_{k=1}^{n-1} \frac{1}{l_{n-1}} kl + 2 \sum_{l_2=2}^{n-4} l_2 + 4 \sum_{l_3=2}^{n-6} l_2 + \ldots + (n-4) \sum_{l_l=2}^{n-4} l_l
\]

Illustrations:

\[
\begin{align*}
\text{n=8, } WW(G) &= 96 \\
\text{n=10, } WW(G) &= 205
\end{align*}
\]

REFERENCES

4. Sandi Klavzar, A Bird’s eye view of the cut method and a survey of its applications in chemical graph theory.