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First Reciprocal Status Connectivity Index, Harmonic Reciprocal Status Indices and Coindices of Line Graphs

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ABSTRACT

In this paper we have studied the first reciprocal status connectivity index, harmonic reciprocal status indices and coindices of line graphs. Explicit expression for indices of line graphs of graph with diameter less than or equal to two are obtained. Further we obtain bounds for first reciprocal status connectivity index, harmonic reciprocal status indices and coindices of line graphs.

AMS Subject Classification: 05C12.

Keywords: Distance in graph, first reciprocal status connectivity index, harmonic reciprocal status index, line graph.

1. INTRODUCTION

In graph theory, the line graph of an undirected graph G is another graph $L(G)$ that represents the adjacencies between edges of G. The name line graph comes from a paper by Harary and Norman (1960) although both Witney (1932) and Krausz (1943) used the construction before[1]. Other terms used for the line graph include the covering graph, the derivative, the edge-to-vertex dual, the conjugate, the representative graph, and the obrazom[1], as well as the edge graph, the interchange graph, the adjoint graph, and the derived graph[2]. Hassler Whitney (1932) proved that with one exceptional case the structure of a connected graph G can be recovered completely from its line graph[3]. Many other properties of line graph follow by translating the properties of the underlying graph from vertices into edges, and by Whitney's theorem the same translation can also be done in the other direction. Line graphs are claw free, and the line graphs of bipartite graphs are perfect. Line graphs are characterized by nine forbidden subgraphs and can be recognized in linear time.

Various extensions of the concept of line graph have been studied, including line graphs of line graphs, line graphs of multigraphs, line graphs of hypergraphs, and line graphs of weighted graphs.

In this paper we are concerned with simple graphs, having no directed or weighted edges, and no self loops. Let G be a connected graph of order n and size m. Let $V(\mathrm{G})$ be the vertex set and $E(\mathrm{G})$ be the edge set of G. The edge joining the vertices u and v is denoted by uv. The degree of a vertex u is the number of edges incident to it and is denoted by $d(\mathrm{u})$. The distance between the vertices u and v is denoted by $d(u,v)$ is the length of the shortest path joining u and v in G.

The line graph [4], of a graph G, denoted by $L(\mathrm{G})$, is the simple graph whose vertices are the edges of G, with two vertices are adjacent in $L(G)$ if and only the corresponding edges have a common end point in G. The complement $\bar{G}$ of a graph G is the graph with vertex set $V(\mathrm{G})$, in which two vertices are adjacent if and only if they are not adjacent in G.

The first and second Zagreb indices of a graph G are defined as[5]

$$M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)] \text{ and } M_2(G) = \sum_{uv \in E(G)} d(u)d(v),$$

where $d(u)$ is the degree of a vertex u in G.

Results on Zagreb indices can be found in[6] and the references cited therein.

The Forgotten topological index, which is vertex degree based graph invariant was studied at the earliest in[7] and defined as,

$$F(G) = \sum_{v \in V(G)} d(v)^3 = \sum_{uv \in E(G)} [d(u)^2 + d(v)^2].$$

The reciprocal status of a vertex $u \in V(\mathrm{G})$ is defined as [9] the sum of reciprocal of its distance from every other vertex in $V(\mathrm{G})$ and is denoted by $rs(\mathrm{u})$. That is

$$rs(u) = \sum_{v \in V(G)} \frac{1}{d(u,v)} \tag{1}$$

First reciprocal status connectivity index $RS_1(\mathrm{G})$ of a connected graph G is defined as[9],

$$RS_1(G) = \sum_{uv \in E(G)} [rs(u) + rs(v)] \ . \tag{2}$$

Harmonic reciprocal status index of a connected graph G is defined as[10],

$$HRS(G) = \sum_{uv \in E(G)} \frac{2}{rs(u) + rs(v)} \ . \tag{3}$$

Harmonic reciprocal status coindex of a connected graph G is defined as[10],

$$\overline{HRS}(G) = \sum_{uv \notin E(G)} \frac{2}{rs(u)+rs(v)}. \quad (4)$$

We need the following results.

Theorem 1.1[11] Let G be a graph with n vertices and m edges. Then,

$$M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1).$$

Theorem 1.2 [11] Let G be a graph with n vertices and m edges and let $L(G)$ be its line graph. Then,

$$M_1(L(G)) = F(G) - 4M_1(G) + 2M_2(G) + 4m.$$

Theorem 1.3[9] Let G be a connected graph with n vertices and m edges and let $\mathrm{diam}(G) = D$. Then

$$\frac{2m}{D}(n-1) + \left(1-\frac{1}{D}\right)M_1(G) \le RS_1(G) \le m(n-1) + \frac{1}{2}M_1(G).$$

Equality holds if and only if $\mathrm{diam}(G) \le 2$.

Corollary 1.4[10] Let G be a connected regular graph of degree r with n vertices and m edges and let $\mathrm{diam}(G) = D$. Then

$$\frac{2m}{(n-1)+r} \le HRS(G) \le \frac{m}{\left(\frac{n-1}{D}\right)+\left(1-\frac{1}{D}\right)r}.$$

Equality on both sides holds if and only if $\mathrm{diam}(G) \le 2$.

Corollary 1.5[10] Let G be a connected regular graph of degree r with n vertices and m edges and let $\mathrm{diam}(G)=D$. Then

$$\frac{n(n-1)-nr}{(n-1)+r} \le \overline{HRS}(G) \le \frac{n(n-1)-nr}{2\left(\left(\frac{n-1}{D}\right)+\left(1-\frac{1}{D}\right)r\right)}.$$

Equality on both sides holds if and only if $\mathrm{diam}(G) \le 2$.

Theorem 1.6[12] For a connected graph G, $diam(L(G)) \le 2$ if and only if none of the three graphs F_1, F_2 *and* F_3 of Fig. 1 is an induced subgraph of G.

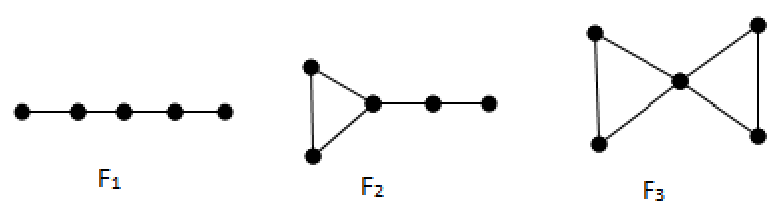

Figure 1

2. FIRST RECIPROCAL STATUS CONNECTIVITY INDEX OF LINE GRAPHS

Theorem2.1. Let G be a connected graph with m edges such that none of the three graphs F_1, F_2 *and* F_3 of Fig. 1 is an induced subgraph of G and $L(G)$ is the line graph. Then

$$RS_1(L(G)) = \frac{1}{2}(m-5)M_1(G) + \frac{1}{2}F(G) + M_2(G) - m(m-3).$$

Proof. From the definition of the line graphs, the number of vertices of $L(G)$ is $n_1 = m$ and the number of edges of $L(G)$ is[3,4]

$$m_1 = \frac{1}{2}\sum_{u \in V(G)} d(u)^2 - m$$

$$= \frac{1}{2}M_1(G) - m.$$

Since from[12], if $diam(G) \leq 2$ and none of the graphs F_1, F_2, F_3 of Fig. 1 is an induced subgraph of G, $diam(L(G)) \leq 2.$ Therefore, by Theorem 1.3

$$RS_1(L(G)) = m_1(n_1 - 1) + \frac{1}{2}M_1(L(G))$$

$$= (m-1)\left[\frac{1}{2}M_1(G) - m\right] + \frac{1}{2}M_1(L(G)).$$

Substituting $M_1(L(G))$ from Theorem 1.2 in the above equation we get

$$RS_1(L(G)) = \frac{1}{2}(m-5)M_1(G) + \frac{1}{2}F(G) + M_2(G) - m(m-3).$$

The following corollary directly follows from the Theorem 2.1.

Corollary 2.2. Let G be a connected regular graph of degree r on n vertices and m edges. Let none of three graphs F_1, F_2 *and* F_3 of Fig. 1 is an induced subgraph of G. Then

$$RS_1(L(G)) = mr(m-5) + 2mr^2 - m(m-3).$$

Proof. For regular graph G of degree r with m vertices,

$M_1(G) = 2mr,\ M_2(G) = mr^2$ *and* $F(G) = 2mr^2$.

Substituting these values in Theorem 2.1 we get the result.

Proposition 2.3 The first reciprocal status connectivity index of line graph of complete bipartite graph $K_{p,q}$ is

$$RS_1\ L\ K_{p,q} = \frac{pq}{2}\left[(p+q)(pq-5) + p^2 + q^2 + 6\right].$$

Proof. The graph $K_{p,q}$, has $n = p+q$ vertices and $m = pq$ edges. Also $diam(K_{p,q}) \leq 2$. The vertex set $V(K_{p,q})$ can be partitioned into two sets V_1 *and* V_2 such that for every edge uv of $K_{p,q}$, the vertex $u \in V_1$ *and* $v \in V_2$, *where* $|V_1| = p$ and $|V_2| = q$. Therefore $d(u) = q$ and $d(v) = p$ and hence $M_1(K_{p,q}) = pq(p+q)$, $M_2(K_{p,q}) = (pq)^2$ *and* $F(K_{p,q}) = pq(p^2+q^2)$. From Theorem 2.1 we get,

$$\begin{aligned} RS_1(L(K_{p,q})) &= \frac{1}{2}(m-5)M_1(K_{p,q}) + \frac{1}{2}F(K_{p,q}) + M_1(K_{p,q}) - m(m-3) \\ &= \frac{1}{2}(m-5)pq(p+q) + \frac{1}{2}pq(p^2+q^2) + (pq)^2 - m(m-3) \\ &= \frac{pq}{2}\left[(p+q)(pq-5) + p^2 + q^2 + 6\right]. \end{aligned}$$

Proposition 2.4 The first reciprocal status connectivity index of line graph of a cycle C_n on n vertices is

$$RS_1(L(C_n)) = \begin{cases} 4\left[n\sum_{i=1}^{\frac{n-2}{2}}\frac{1}{i} + 1\right] & if \;\; n \;\; is \;\; even \\ 4n\sum_{i=1}^{\frac{n-2}{2}}\frac{1}{i} & if \;\; n \;\; is \;\; odd. \end{cases}$$

Proof. As we know that $L(C_n) = C_n$. Therefore from[9] we get the result.

Theorem 2.5 If G is any connected graph with n vertices, m edges, maximum degree $\Delta(G)$ *and* $diam(G) = D$. Then the first reciprocal status connectivity index of its line graph is,

$$RS_1(L(G)) \geq 2\left[\frac{1}{2}M_1(G) - m\right]\frac{m-1}{(m-2)(\Delta(G)-1)+1} + \left(1 - \frac{1}{(m-2)(\Delta(G)-1)+1}\right)(F(G) - 4M_1(G) + M_2(G) + 4m).$$

and

$$RS_1(L(G)) \leq (m-1)\left(\frac{1}{2}M_1(G) - m\right) + \frac{1}{2}(F(G) - 4M_1(G) + 2M_2(G) + 4m).$$

Proof. We know from Theorem 1.3 the first reciprocal status connectivity index of graph G is

$$\frac{2m}{D}(\mathrm{n}-1)+\left(1-\frac{1}{D}\right)M_1(\mathrm{G})\le \mathrm{RS}_1(\mathrm{G}). \tag{5}$$

Now for any graph G, the number of vertices of its line graph $L(\mathrm{G})$ is $n_1=m$ and the number of edges is $m_1=\frac{1}{2}\sum_{i=1}^{n}d_i^{\,2}-m.$ Therefore, from Eq. (5) we have

$$\begin{aligned} RS_1\left(L\left(G\right)\right) &\ge \frac{2m_1(\mathrm{m}-1)}{D_L}+\left(1-\frac{1}{D_L}M_1\left(L\left(G\right)\right)\right)\\ &\ge 2\left[\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right]\frac{m-1}{D_L}+\left(1-\frac{1}{D_L}\right)M_1\left(L(\mathrm{G})\right). \end{aligned} \tag{6}$$

Where, D_L is the diameter of $L(\mathrm{G})$. For any graph G of order *n*, $diam(\mathrm{G})\le \mathrm{n}-\Delta(\mathrm{G})+1$ and hence

$$\frac{1}{D_L}\ge\frac{1}{m-\Delta\left(L(\mathrm{G})\right)+1}. \tag{7}$$

Also for any graph G,

$$\frac{1}{\Delta\left(L(\mathrm{G})\right)}\le\frac{1}{2\left(\Delta(\mathrm{G})-1\right)}. \tag{8}$$

Substituting Eq. (7) and Eq. (8) in Eq. (6) we get the following.

$$\begin{aligned} RS_1\left(L(\mathrm{G})\right) &\ge 2\left[\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right]\frac{m-1}{m-2\left(\Delta(\mathrm{G})-1\right)+1}\\ &+\left(1-\frac{1}{m-2\left(\Delta(\mathrm{G})-1\right)+1}\right)\left(F(\mathrm{G})-4\mathrm{M}_1(\mathrm{G})+2\mathrm{M}_2(\mathrm{G})+4\mathrm{m}\right). \end{aligned}$$

We know from Theorem 1.3 the first reciprocal status connectivity index of G is

$$RS_1(\mathrm{G})\le \mathrm{m}(\mathrm{n}-1)+\frac{1}{2}M_1(\mathrm{G}). \tag{9}$$

Now for any graph G, the number of vertices of its line graph $L(\mathrm{G})$ is $n_1=m$ and the number of edges is $m_1=\frac{1}{2}\sum_{i=1}^{n}d_i^{\,2}-m.$ Therefore, from Eq. (9) we have

$$RS_1(L(G))\le m_1(m-1)+\frac{1}{2}M_1(L(G))$$

$$RS_1(L(G))\le (m-1)\left(\frac{1}{2}M_1(G)-m\right)+\frac{1}{2}(F(G)-4M_1(G)+2M_2(G)+4m).$$

Theorem 2.6. Let G be a graph with n vertices and m edges, whose line graph $L(\mathrm{G})$ has $diam\ \ L(\mathrm{G})\ >3.$ Then

$$RS_1\ \ \overline{L(\mathrm{G})}\ =\frac{M_1(\mathrm{G})}{2}(\mathrm{m}+3)+\frac{1}{2}F(\mathrm{G})+\mathrm{M}_2(\mathrm{G})+\mathrm{m}\left[\frac{1}{2}m^2-2m+\frac{7}{2}\right].$$

Proof. Let $\overline{L(G)}$ be the complement of L(G). By Theorem 1.3, we have

$$RS_1(\mathrm{G})\le \mathrm{m}(\mathrm{n}-1)+\frac{1}{2}M_1(\mathrm{G}).$$

Equality holds for graphs with diam(G) ≤ 2. Since if diam(L(G)) > 3, then diam($\overline{L(G)}$) ≤ 2. Therefore if n_1 and m_1 are the number of vertices and number of edges of $\overline{L(G)}$ respectively, then

$$RS_1(\overline{L(\mathrm{G})})=\mathrm{m}_1(\mathrm{n}_1-1)+\frac{1}{2}M_1(\overline{L(\mathrm{G})})$$

$$=\left[\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right](\mathrm{m}-1)+\frac{1}{2}M_1(\overline{L(\mathrm{G})}).$$

Now, from the Theorem 1.1 we have,

$$RS_1(\overline{L(\mathrm{G})})=\left[\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right](\mathrm{m}-1)+\frac{1}{2}\left[M_1(\mathrm{L(G)})+\mathrm{m}(\mathrm{m}-1)^2-4m_1(\mathrm{m}-1)\right]$$

$$=\left[\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right](\mathrm{m}-1)+\frac{1}{2}\left[M_1(\mathrm{L(G)})+\mathrm{m}(\mathrm{m}-1)^2-4\left(\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right)(\mathrm{m}-1)\right]$$

$$=\left[\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right](\mathrm{m}-1)+\frac{1}{2}\left[\begin{array}{l}F(\mathrm{G})-4\,\mathrm{M}_1(\mathrm{G})+2\,\mathrm{M}_2(\mathrm{G})+4\,\mathrm{m}+m\ \ (\mathrm{m}-1)^2\\ -4\left(\frac{1}{2}M_1(\mathrm{G})-\mathrm{m}\right)(\mathrm{m}-1)\end{array}\right]$$

$$=\frac{M_1(\mathrm{G})}{2}(\mathrm{m}+3)+\frac{1}{2}F(\mathrm{G})+\mathrm{M}_2(\mathrm{G})+\mathrm{m}\left[\frac{1}{2}m^2-2m+\frac{7}{2}\right].$$

3. HARMONIC RECIPROCAL STATUS INDICES AND COINDICES OF LINE GRAPHS

Theorem 3.1. Let G be a connected r-regular graph on n vertices and m edges with diam(G) ≤ 2. If none of F_i, i = 1, 2, 3 of Fig. 1 is an induced subgraph of G, then

$$HRS\ \ L(\mathrm{G})\ =\frac{nr^2-2m}{m+2r-3}$$

and

$$\overline{HRS}\ \ L(\mathrm{G})\ = \frac{m(m-2r+1)}{(\mathrm{m}+2\,\mathrm{r}-3)}.$$

Proof. From the definition of the line graphs [4], the number of vertices of $L(\mathrm{G})$ is $n_1 = m$ and the number of edges is $m_1 = -m + \frac{1}{2}\sum_{i=1}^{n} d_i^{\,2} = -m + \frac{1}{2}nr^2$. The regularity of $L(G)$ is $r_1 = 2r - 2$. Therefore by Corollary 1.4

$$HRS(L(G)) = \frac{2m_1}{(n_1-1)+r_1} = \frac{-2m+nr^2}{m+2r-3}.$$

Also by Theorem 1.5 we have

$$\overline{HRS}(L(G)) = \frac{2n_1(n_1-1)-n_1r_1}{(n_1-1)+r_1} = \frac{m(m-2r+1)}{m+2r-3}.$$

Theorem 3.2. Let G be any connected graph with n vertices, m edges and maximum degree Δ. Then

$$HRS(L(G)) \geq \frac{-2m+M_1(G)}{2\Delta-3+m}$$

and

$$\overline{HRS}(L(G)) \geq \frac{m^2+m-M_1(G)}{2\Delta-3+m}.$$

Proof. From the definition of the line graphs [4], the number of vertices of $L(\mathrm{G})$ is $n_1 = m$. Consider an edge $e = uv \in E(\mathrm{G})$, which is adjacent to $d(\mathrm{u})+\mathrm{d}(\mathrm{v})-2 = d(\mathrm{e})$ edges at u and v taken together in $L(\mathrm{G})$. Hence the edge e is not adjacent to remaining $m-1-d(\mathrm{e})$ of G. In $L(\mathrm{G})$ the distance between e and the remaining $m-1-d(\mathrm{e})$ vertices is at least 2. Hence for any graph G, $rs(e)$ in $L(G)$ is

$$rs(\mathrm{e}) \leq \ d(\mathrm{u})+\mathrm{d}(\mathrm{v})-2\ + \frac{1}{2}\ m-1-\ d(\mathrm{u})+\mathrm{d}(\mathrm{v})-2 \ \ .$$

Since Δ is the maximum degree, $d(\mathrm{u})+\mathrm{d}(\mathrm{v}) \leq 2\Delta$. Therefore

$$rs(\mathrm{e}) \leq \frac{1}{2}\ \ 2\Delta-3+m\ \ .$$

Now suppose e and f are two vertices of $L(\mathrm{G})$, then

$$rs(\mathrm{e})+\mathrm{rs}(\mathrm{f}) \leq \ 2\Delta\ \ -3+m.$$

Now, from Eq. (3) we have,

$$HRS(L(G)) = \sum_{ef \in E(L(G))} \frac{2}{rs(e)+rs(f)} \geq \sum_{ef \in E(L(G))} \frac{2}{2\Delta - 3 + m}$$

$$= \left(-m + \frac{1}{2}\sum_{i=1}^{n} {d_i}^2\right)\left(\frac{2}{2\Delta - 3 + m}\right) = \frac{-2m + M_1(G)}{2\Delta - 3 + m}.$$

Now from Eq. (4)

$$\overline{HRS}(L(G)) = \sum_{ef \notin E(L(G))} \frac{2}{rs(e)+rs(f)} \geq \sum_{ef \notin E(L(G))} \frac{2}{2\Delta - 3 + m}$$

$$= \left(\frac{n_1(n_1-1)}{2} - m_1\right)\left(\frac{2}{2\Delta - 3 + m}\right) = \left(\frac{m(m-1)}{2} + m - \frac{1}{2}\sum_{i=1}^{n} {d_i}^2\right)\left(\frac{2}{2\Delta - 3 + m}\right)$$

$$= \frac{m^2 + m - M_1(G)}{2\Delta - 3 + m}.$$

4. CONCLUSION

The first reciprocal status connectivity indices, Harmonic reciprocal status indices and coindices of line graph is obtained. Exact formula for any graph with diameter less than or equal to 2 is obtained. Further we obtain bounds for first reciprocal status connectivity index, Harmonic reciprocal status indices and coindices of line graphs.

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