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Plus Weighted Finite State Automaton

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ABSTRACT

This paper introduces a plus weighted finite state automaton (pwfa) with weighted regular language. A plus weighted finite state automaton assigns a plus weighted values in which there is a unique weighted transition on an input symbol is considered. It is shown that for a given incomplete pwfa there exists an equivalent complete pwfa with same weighted regular language. Some Closure properties on pwfa are analyzed.

Keywords: Plus weighted finite state automaton, recognizability, weighted regular language.

1. INTRODUCTION

Research in formal language theory, which is one of the fundamental areas of theoretical computer science, started in the year 1950's, when Noman Chomsky in his study on grammars and grammatical structure of language, proposed a mathematical model of a grammar. Finite automata plays a crucial role in the theory of programming languages, compiler constructions, switching circuit designing, computer controllers, neuron net, text editor and lexical analyzer[1].

Historically, weighted automata were introduced in the seminal paper by Schutzenberger[5]. A close relationship to probabilistic automata was mutually influential in the beginning. For the domain of weights and their computations the algebraic structure of semirings proved to be fruitful. This soon leads to a rich mathematical theory including applications for purely language theoretic questions as well as practical applications in digital image compression and algorithm for natural image processing. Excellent treatments of this are provided by the books and the surveys in the recent handbook.

In this paper language related to plus weighted finite state automaton is introduced. Some of the closure properties of plus weighted finite state automata languages are considered.

2. RECOGNIZABILITY OF pwfa

This section deals with a relevant definition of pwfa and recognizalility of pwfa are analyzed using suitable theorem and examples.

Definition 2.1.[3] A plus weighted finite state automaton (pwfa) is a sextuple $P = (\mathrm{Q}, \Sigma, \mathrm{W}, \mu, \pi, \eta)$, where

(i) Q is a finite non-empty set of states.

(ii) Σ is a finite non-empty set of input symbols.

(iii) W is a weighting space.
i.e., weighting space $W = ([0,\infty), +, \cdot)$, $+$ and $\cdot$ are usual addition and multiplication.

(iv) the weighted subset $\mu : \mathrm{Q} \times \Sigma \times \mathrm{Q} \rightarrow [0,\infty)$ is a function called the weighted transition function.

(v) π is a weighted subset of Q.
i.e., $\pi : \mathrm{Q} \rightarrow [0,\infty)$ called the weighted subset of initial states.

(vi) η is a weighted subset of finial states.
i.e., $\eta : \mathrm{Q} \rightarrow [0,\infty)$ called the weighted subset of final states.

Definition 2.2. Let $P = (\mathrm{Q}, \Sigma, \mathrm{W}, \mu, \pi, \eta)$ be a pwfa, the extended weighted function for P is the weigted subset $\mu^* : \mathrm{Q} \times \Sigma^* \times \mathrm{Q} \rightarrow [0,\infty)$ with usual addition + and multiplication $\cdot$ has been defined as follows: $\forall p, q, q' \in Q, a \in \Sigma, x \in \Sigma^*$

$$\mu^*(p, \lambda, q) = \begin{cases} 1, & if \quad p = q \\ 0, & if \quad p \neq q \end{cases}$$

$$\mu^*(p, \mathrm{xa}, q') = \sum_{q \in Q} \mu^*(p, \mathrm{x}, q) \cdot \mu(\mathrm{q}, \mathrm{a}, q')$$

Definition 2.3. Let $P = (\mathrm{Q}, \Sigma, \mathrm{W}, \mu, \pi, \eta)$ be a pwfa. Let $x \in \Sigma^*$. Then x is said to be recognized by P if

$$L(x) = \sum_{p,q \in Q} \pi(p) \cdot \mu^*(p, x, q) \cdot \eta(q) > 0$$

$$= \sum_{p,q \in Q} \pi(p) \cdot \left\{ \sum_{r \in \mathrm{Q}} \mu^*(p, \mathrm{y}, \mathrm{r}) \cdot \mu(\mathrm{r}, \mathrm{a}, q) \right\} \cdot \eta(q) \text{ where } x = ya$$

Lemma 2.4. Let $P = (\mathrm{Q}, \Sigma, \mathrm{W}, \mu, \pi, \eta)$ be a pwfa. Let $x \in \Sigma^*$.Then x is recognized if and only if there exist $p, q \in Q$ such that $w(x) = \pi(p) \cdot \mu^*(p, x, q) \cdot \eta(q) > 0$.

Theorem 2.5. Let $P = (Q, \Sigma, W, \mu, \pi, \eta)$ be a pwfa. Then $\forall p, q \in Q$ and $x, y \in \Sigma^*$

$$\mu^*(p, xy, q) = \sum_{r \in Q} \mu^*(p, x, r) \cdot \mu^*(r, y, q).$$

Proof: Let $p, q \in Q$ and $x, y \in \Sigma^*$.

We prove the result by induction on $|y| = n.$ If $n = 0$, the result is obvious.

Suppose the result is true $\forall u \in \Sigma^*$, such that $|u| \le (n-1), n > 0$,Now

$$\mu^*(p, xy, q) = \mu^*(p, xua, q) = \sum_{r \in Q} \mu^*(p, xu, r) \cdot \mu(r, a, q)$$

$$= \sum_{s \in Q} \mu^*(p, x, s) \cdot \left\{ \sum_{r \in Q} \mu^*(s, u, r) \cdot \mu(r, a, q) \right\}$$

$$= \sum_{s \in Q} \mu^*(p, x, s) \cdot \mu^*(s, y, q)$$

Thus the result is true for $|y| = n$ and hence the result.

Example 2.6. Let $P = (Q, \Sigma, W, \mu, \pi, \eta)$ be a pwfa, where $Q = \{q_1, q_2, q_3\}$, $\Sigma = \{a, b\}$,

$\mu : Q \times \Sigma \times Q \to [0, \infty)$ is defined as follows:

$\mu(q_1, a, q_1) = 2$, $\mu(q_1, b, q_2) = 1$,

$\mu(q_1, a, q_2) = 2$, $\mu(q_2, a, q_3) = 3$,

$\mu(q_1, b, q_1) = 3$.

We omit the weight values which are zero.

$\pi : Q \to [0, \infty)$ defined by $\pi(q_1) = 3$, $\pi(q_2) = \pi(q_3) = 0$

$\eta : Q \to [0, \infty)$ defined by $\eta(q_3) = 3$, $\eta(q_1) = \eta(q_2) = 0$

The transition diagram is shown below:

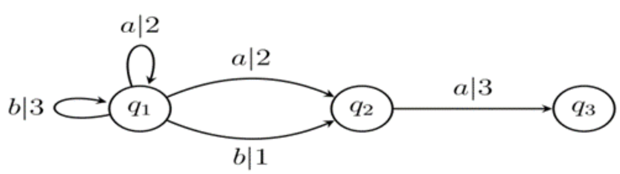

Figure 2.1: Example for plus weighted finite state automaton.

The language accepted by P is a weighted subset $L_P : \Sigma^* \to [0, \infty)$ such that

$$L_P(x) = \begin{cases} w_1, & w_1 \ge 54 \quad \text{if} \quad x \in \{a, b\}^* aa \\ w_2, & w_2 \ge 27 \quad \text{if} \quad x \in \{a, b\}^* ba \\ 0, & \text{otherwise} \end{cases}$$

Here a first attempt is made in proving that the weighted regular language accepted by incomplete pwfa and complete weighted regular language are same.

Definition 2.7. Let $P=(\mathrm{Q},\sum,\mathrm{W},\mu,\pi,\eta)$ be a pwfa. P is called complete if $\forall p\in Q, a\in\sum$
there exists $q\in Q$ such that $\mu(\mathrm{p},a,q)>0$.

Theorem 2.8. Let $P=(\mathrm{Q},\sum,\mathrm{W},\mu,\pi,\eta)$ be an incomplete pwfa, then there exists a pwfa P^c which is the completion of P such that the weighted regular language accepted by P and P^c are equal.

Proof: Let $P=(\mathrm{Q},\sum,\mathrm{W},\mu,\pi,\eta)$ be an incomplete pwfa and the weighted language accepted by it be L. Let $Q^c=Q\cup\{d\}$ where d is a new state such that $d\notin Q$. $\forall p\in Q$, let $0<m_p<\infty$ and let $0<m<\infty$. Define

$P^c=(\mathrm{Q}^c,\sum,\mathrm{W},\mu,\pi,\eta)$ where $\mu^c:\mathrm{Q}^c\times\sum\times\mathrm{Q}^c\to[0,\infty)$ is defined as follows: $\forall p,q\in Q, a\in\sum$

(i) $\mu^c(p,a,q)=\mu(p,a,q), \quad if \quad \mu(p,a,q)>0$

(ii) $\mu^c(p,a,d)=\begin{cases} m_p, & if \quad \sum_{q\in Q}\mu(p,a,q)=0 \\ 0, & if \quad \sum_{q\in Q}\mu(p,a,q)>0 \end{cases}$

(iii) $\mu^c(\mathrm{d},a,\mathrm{p})=\begin{cases} m, & if \quad p=d \\ 0, & if \quad p\neq d \end{cases}$

$\pi^c:Q^c\to[0,\infty)$ is defined by

$$\pi^c(p)=\begin{cases} \pi(\mathrm{p}), & if \quad p\in Q \\ 0, & if \quad p\notin Q \end{cases}$$

$\eta^c:Q^c\to[0,\infty)$ is defined by

$$\eta^c(p)=\begin{cases} \eta(p), & if \quad p\in Q \\ 0, & if \quad p\notin Q \end{cases}$$

Clearly P^c is complete pwfa. Let L_1 be a weighted language accepted by P^c. Next we prove $L=L_1$.i.e., to prove $L(x)=L_1(x)$ for all $x\in\sum^*$

Case (i) : $\mathrm{L}(\mathrm{x})=0$

Implies that $\pi(\mathrm{p})=0$ $\forall p\in Q$ or $\mu^*(\mathrm{p},\mathrm{x},q)=0$, $\eta(q)=0$.

If $\pi(\mathrm{p}) = 0 \;\; \forall p \in Q$ then $\pi^c(p) = 0 \;\; \forall p \in Q^c$. Therefore, $L_1(x) = 0$

If $\eta(q) = 0$, $\forall q \in Q$ then $\eta^c(q) = 0$, $\forall q \in Q^c$. Therefore, $L_1(x) = 0$.

Suppose $\pi(p) \neq 0$, $\eta(q) \neq 0$ then $\mu^*(\mathrm{p}, \mathrm{x}, q) = 0$.

Let $x = a_1 a_2 ... a_n$, implies for some a_k, $1 \leq k \leq n$, there is no move in P.Let j be the smallest integer such that $1 \leq j \leq n$, there is no move in P on a_j from p_j. From the construction of P^c, the automaton enters into a dead state d on a_j with value m_{p_j} .Then after P^c halts at d by leading the remaining input symbols. But $\pi^c(\mathrm{d}) = 0$ .Therefore $L_1(x) = 0$. Similarly converse can be proved. Thus $L(x) = L_1(x)$.

Case (ii): $L(x) > 0$

Since Q is finite and each term is non-negative there exist $p, q \in Q$ such that $w(x) = \pi(p) \cdot \mu^*(\mathrm{p}, \mathrm{x}, q) \cdot \eta(q)$

$\mathrm{w}(x) > 0$ implies $\pi(p) > 0$, $\mu^*(\mathrm{p}, \mathrm{x}, q) > 0$, $\eta(q) > 0$.

$\pi(p) > 0$ implies $\pi^c(p) = \pi(p)$, $p \in Q^c$, $\eta(q) > 0$ implies $\eta^c(q) = \eta(q)$, $q \in Q^c$, $\mu^*(\mathrm{p}, \mathrm{x}, q) > 0$ from the definition of P^c, $\mu^{c*}(\mathrm{p}, \mathrm{x}, q) = \mu(\mathrm{p}, \mathrm{x}, q)$.

Thus, $w(x) = \pi^c(p) \cdot \mu^{c*}(\mathrm{p}, \mathrm{x}, q) \cdot \eta^c(q)$. i.e., $w(x) = w^c(x)$.

Then, $\sum_{p,q \in Q} \pi(p) \cdot \mu^*(\mathrm{p}, \mathrm{x}, q) \cdot \eta(q) = \sum_{p,q \in Q^c} \pi^c(p) \cdot \mu^{c*}(\mathrm{p}, \mathrm{x}, q) \cdot \eta^c(q)$.

i.e., $L(x) = L_1(x)$. Similarly we can prove $L_1(x) = L(x)$.

Example 2.9. Consider an incomplete pwfa $P = (\mathrm{Q}, \Sigma, \mathrm{W}, \mu, \pi, \eta)$ where, $\Sigma = \{0,1\}$ $\mu : \mathrm{Q} \times \Sigma \times \mathrm{Q} \to [0, \infty)$ is defined as follows:

$\mu(\mathrm{q}_1, 0, q_1) = 2.4$, $\quad \mu(\mathrm{q}_2, 0, q_3) = 2.3$, $\quad \mu(\mathrm{q}_3, 0, q_3) = 2.3$,

$\mu(\mathrm{q}_1, 1, q_1) = 2.1$, $\quad \mu(\mathrm{q}_1, 1, q_2) = 2$, $\quad \mu(\mathrm{q}_3, 1, q_3) = 2.4$

and $\pi(q_1) = 2.3, \pi(q_2) = 0.5;$ $\quad \eta(q_2) = 2.1, \eta(q_3) = 2.$

The transition diagram is shown below:

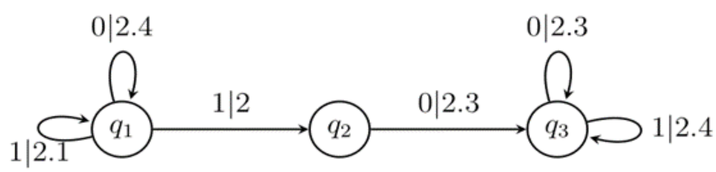

Figure 2.2: Example for an incomplete plus weighted finite state automaton.

The language accepted by P is a weighted subset $L : \Sigma^* \to [0, \infty)$ such that

$$L(x)=\begin{cases} w_1, & w_1 \geq 9.66 \quad if \quad x \in \{0,1\}^*1 \\ w_2, & w_2 \geq 21.16 \quad if \quad x \in \{0,1\}^* 10\{0,1\}^* \\ \mathrm{w}_3, & w_3 \geq 2.3 \quad if \quad x \in 0\{0,1\}^* \\ 0, & otherwise \end{cases}$$

Define $P^c = (\mathrm{Q}^c, \Sigma, \mathrm{W}, \mu^c, \pi^c, \eta^c)$ where $\mathrm{Q}^c = Q \cup \{q_4\}$, $\Sigma = \{0,1\}$,

$\mu^c : \mathrm{Q}^c \times \Sigma \times \mathrm{Q}^c \to [0,\infty)$

is defined as follows:

$\mu^c(\mathrm{q}_1, 0, q_1) = 2.4$, $\mu^c(\mathrm{q}_1, 1, q_1) = 2.1$, $\mu^c(\mathrm{q}_2, 0, q_3) = 2.3$,

$\mu^c(\mathrm{q}_1, 1, q_2) = 2$, $\mu^c(\mathrm{q}_3, 0, q_3) = 2.3$, $\mu^c(\mathrm{q}_3, 1, q_3) = 2.4$,

$\mu^c(\mathrm{q}_4, 0, q_4) = 2.2$, $\mu^c(\mathrm{q}_2, 1, q_4) = 1.5$, $\mu^c(\mathrm{q}_4, 1, q_4) = 2.2$.

$\pi^c : Q^c \to [0,\infty)$ is defined by $\pi^c(q_1) = 2.3$, $\pi^c(q_2) = 0.5$

$\eta^c : Q^c \to [0,\infty)$ is defined by $\eta^c(q_2) = 2.1$, $\eta^c(q_3) = 2$.

The language accepted by P^c is a weighted subset $L_1 : \Sigma^* \to [0,\infty)$ such that

$$L_1(x)=\begin{cases} w_1, & w_1 \geq 9.66 \quad if \quad x \in \{0,1\}^*1 \\ w_2, & w_2 \geq 21.16 \quad if \quad x \in \{0,1\}^* 10\{0,1\}^* \\ \mathrm{w}_3, & w_3 \geq 2.3 \quad if \quad x \in 0\{0,1\}^* \\ 0, & otherwise \end{cases}$$

The transition diagram is shown below:

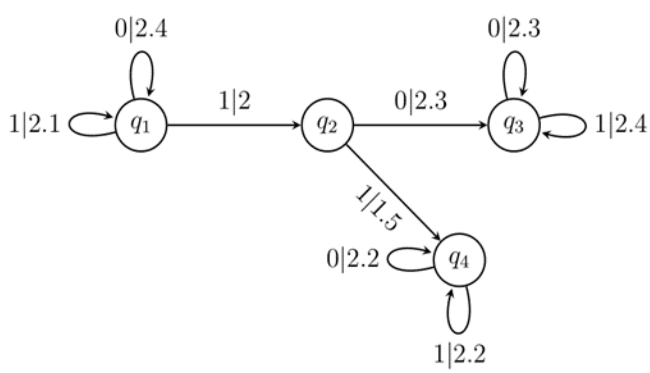

Figure 2.3: Example for complete plus weighted finite state automaton.

Clearly the weighted regular language accepted by P^c is L.

3. CLOSURE PROPERTIES OF WEIGHTED REGULAR LANGUAGES

Some closure properties such as union, intersection, concatenation and Kleene's closure on weighted regular languages are discussed with examples.

Theorem 3.1. Let $P_1 = (Q_1, \Sigma, W, \mu_1, \pi_1, \eta_1)$ and $P_2 = (Q_2, \Sigma, W, \mu_2, \pi_2, \eta_2)$ be two pwfa'swith weighted regular languages L_1 and L_2 respectively. Then L is a weighted regular language accepted by $P_1 \cup P_2$ where $L = L_1 \cup L_2$ and $L(x) = L_1(x) + L_2(x)$.

Proof: Let union of P_1 and P_2 is a pwfa $P = P_1 \cup P_2 = (Q', \Sigma, W, \mu, \pi, \eta)$ where $Q' = Q_1 \cup Q_2 \cup \{q_0\}$, $\mu : Q' \times \Sigma \times Q' \to [0, \infty)$ is defined as follows: $\forall\, p, q \in Q$

(i) $\mu(p, a, q) = \mu_1(p, a, q), \forall\, p, q \in Q_1, a \in \Sigma$

(ii) $\mu(p, a, q) = \mu_2(p, a, q), \forall\, p, q \in Q_2, a \in \Sigma$

(iii) For $p, q \in Q_1$, if $r \in Q_1$, $\pi_1(p) > 0$ and $\mu_1(p, a, r) > 0$ $\quad \exists q_0 \notin Q$ such that $\mu(q_0, a, r) = \pi_1(p) \cdot \mu_1(p, a, r)$

(iv) For $p, q \in Q_2$, if $r \in Q_2$, $\pi_2(p) > 0$ and $\mu_2(p, a, r) > 0$, $\quad \exists q_0 \notin Q$ such that $\mu(q_0, a, r) = \pi_2(p) \cdot \mu_2(p, a, r)$

$\pi : Q \to [0, \infty)$ is defined by

$$\pi(p) = \begin{cases} \pi_1(p), & if \quad p \in Q_1 \\ \pi_2(p), & if \quad p \in Q_2 \end{cases}$$

$\eta : Q \to [0, \infty)$ is defined by

$$\eta(p) = \begin{cases} \eta_1(p), & if \quad p \in Q_1 \\ \eta_2(p), & if \quad p \in Q_2 \end{cases}$$

From the definition of P ,we have for all $x \in \Sigma^*$, then if $q \in Q_1$ and $q_0 \in Q' \setminus Q$

$$\mu^*(p, x, q) = \begin{cases} \mu_1^*(p, x, q), & if \quad p, q \in Q_1 \\ \mu_2^*(p, x, q), & if \quad p, q \in Q_2 \\ 0, & otherwise \end{cases}$$

$$\mu^*(q_0, x, q) = \sum_{r \in Q_1} \mu(q_0, a, r) \cdot \mu^*(r, y, q) + \sum_{r \in Q_2} \mu(q_0, a, r) \cdot \mu^*(r, y, q) \quad (1)$$

$$= \pi_1(p) \cdot \mu^*(p, x, q)$$

Let $x \in \Sigma^*$, then if $q \in Q_2$ and $q_0 \in Q' \setminus Q$ then

$$\mu^*(q_0, x, q) = \pi_2(p) \cdot \mu^*(p, x, q) \quad (2)$$

Now,

$$\mathrm{L(x)} = \sum_{(q_0,\mathrm{q})\in \mathrm{Q}'} \pi(q_0)\cdot \mu^*(\mathrm{q}_0, x, q)\cdot \eta(q)$$

$$= \sum_{\mathrm{q}\in \mathrm{Q}_1} \pi(\mathrm{q}_0)\cdot \pi_1(\mathrm{p})\cdot \mu_1^*(\mathrm{p}, x, q)\cdot \eta_1(q) + \sum_{\mathrm{q}\in \mathrm{Q}_2} \pi(\mathrm{q}_0)\cdot \pi_2(\mathrm{p})\cdot \mu_2^*(\mathrm{p}, x, q)\cdot \eta_2(q)$$

(from (1) and (2))

$$= \sum_{\mathrm{p,q}\in \mathrm{Q}_1} \pi(\mathrm{q}_0)\cdot \pi_1(\mathrm{p})\cdot \mu_1^*(\mathrm{p}, x, q)\cdot \eta_1(q) + \sum_{\mathrm{p,q}\in \mathrm{Q}_2} \pi(\mathrm{q}_0)\cdot \pi_2(\mathrm{p})\cdot \mu_2^*(\mathrm{p}, x, q)\cdot \eta_2(q)$$

(since $\pi(\mathrm{q}_0) = 1$)

$= L_1(x) + L_2(x).$ Hence $L = L_1 \cup L_2$.

Definition 3.2. Let $\mathrm{P}_1 = (\mathrm{Q}_1, \Sigma, \mathrm{W}, \mu_1, \pi_1, \eta_1)$ and $\mathrm{P}_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \pi_2, \eta_2)$ be two pwfa's $Q_1 \cap Q_2 = \varnothing$. Then the weighted finite state automata

$\mathrm{P}_1 \cup P_2 = (Q_1 \times Q_2, \Sigma, \mathrm{W}, \mu_1 \times \mu_2, \pi_1 \times \pi_2, \eta_1 \times \eta_2)$ where

$\mu_1 \times \mu_2 : (Q_1 \times Q_2) \times \Sigma \times (Q_1 \times Q_2) \to [0, \infty)$

is defined by $(\mu_1 \times \mu_2)((p_1, p_2), a, (\mathrm{q}_1, q_2)) = \mu_1(\mathrm{p}_1, \mathrm{a}, q_1)\cdot \mu_2(\mathrm{p}_2, \mathrm{a}, q_2)$,

$(\pi_1 \times \pi_2) : Q_1 \times Q_2 \to [0, \infty)$ is defined by $(\pi_1 \times \pi_2)(p_1, p_2) = \pi_1(p_1)\cdot \pi_2(p_2)$,

$(\eta_1 \times \eta_2) : Q_1 \times Q_2 \to [0, \infty)$ is defined by $(\eta_1 \times \eta_2)(p_1, p_2) = \eta_1(p_1)\cdot \eta_2(p_2)$.

Lemma 3.3. Let $\mathrm{P}_1 = (\mathrm{Q}_1, \Sigma, \mathrm{W}, \mu_1, \pi_1, \eta_1)$ and $\mathrm{P}_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \pi_2, \eta_2)$ be two pwfa's. Then,

$(\mu_1 \times \mu_2)^*((p_1, p_2), \mathrm{x}, (\mathrm{q}_1, q_2)) = \mu_1^*(\mathrm{p}_1, \mathrm{x}, q_1)\cdot \mu_2^*(\mathrm{p}_2, \mathrm{x}, q_2)$, $\forall\, \mathrm{x} \in \Sigma^*$,

$(p_1, p_2), (\mathrm{q}_1, q_2) \in Q_1 \times Q_2$.

Theorem 3.4. Let $\mathrm{P}_1 = (\mathrm{Q}_1, \Sigma, \mathrm{W}, \mu_1, \pi_1, \eta_1)$ and $\mathrm{P}_2 = (\mathrm{Q}_2, \Sigma, \mathrm{W}, \mu_2, \pi_2, \eta_2)$ be two pwfa's with L_1 and L_2 as a weighted regular languages respectively. Then L is a weighted regular language accepted by pwfa P such that $\mathrm{L}_1 \cap L_2$ and $\mathrm{L(x)} = L_1(x)\cdot L_2(x)$.

Proof: Let $\mathrm{P} = \mathrm{P}_1 \cap P_2$ and L be a weighted regular language accepted by P. Let $x \in \Sigma^*$

$$\mathrm{L(x)} = \sum_{(p_1,\mathrm{p}_2),(\mathrm{q}_1,\mathrm{q}_2)\in(Q_1\times Q_2)} \left\{ (\pi_1 \times \pi_2)(p_1, p_2)\cdot(\mu_1 \times \mu_2)^*((p_1, p_2), \mathrm{x}, (\mathrm{q}_1, q_2))\cdot(\eta_1 \times \eta_2)(\mathrm{q}_1, \mathrm{q}_2) \right\}$$

$$= \sum_{(p_1,\mathrm{q}_1\in Q_1)(p_2,\mathrm{q}_2\in Q_2)} \left\{ \pi_1(p_1)\cdot \pi_2(p_2)\cdot \mu_1^*(\mathrm{p}_1, \mathrm{x}, q_1)\cdot \mu_2^*(\mathrm{p}_2, \mathrm{x}, q_2)\cdot \eta_1(\mathrm{q}_1)\cdot \eta_2(\mathrm{q}_2) \right\}$$

$$= \sum_{(p_1,\mathrm{q}_1\in Q_1)} \left\{ \pi_1(p_1)\cdot \mu_1^*(\mathrm{p}_1, \mathrm{x}, q_1)\cdot \eta_1(\mathrm{q}_1) \right\}\cdot \sum_{(p_2,\mathrm{q}_2\in Q_2)} \left\{ \pi_2(p_2)\cdot \mu_2^*(\mathrm{p}_2, \mathrm{x}, q_2)\cdot \eta_2(\mathrm{q}_2) \right\}$$

(by previous lemma)

$= L_1(x)\cdot L_2(x)$. Thus $\mathrm{L(x)} = L_1(x)\cdot L_2(x)$ and $L = L_1 \cap L_2$ for all $\mathrm{x} \in \Sigma^*$.

Example 3.5. Let $P_1 = (Q_1, \Sigma, W, \mu_1, \pi_1, \eta_1)$ be a pwfa, where, $Q_1 = \{q_1, q_2\}$, $\Sigma = \{a, b\}$,
$\mu_1 : Q_1 \times \Sigma \times Q_1 \to [0, \infty)$ is defined as follows:
$\mu_1(q_1, a, q_1) = 1$, $\mu_1(q_1, b, q_1) = 2$, $\mu_1(q_1, a, q_2) = 2$,
and $\pi_1(q_1) = 3, \pi_1(q_2) = 1$; $\eta_1(q_1) = 2, \eta_1(q_2) = 4$.
The transition diagram is shown below:

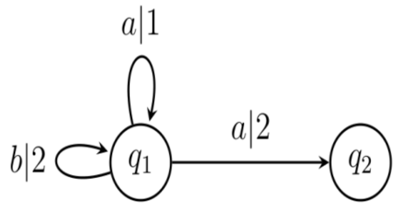

Figure 3.1

The language accepted by P_1 is a weighted subset $L_1 : \Sigma^* \to [0, \infty)$ such that

$$L_1(x) = \begin{cases} w_1, & w_1 \geq 6 \quad if \quad x \in \{a,b\}^* \\ w_2, & w_2 \geq 24 \quad if \quad x \in \{a,b\}^* a \\ 0, & otherwise \end{cases}$$

$$L_1(aa) = 3 \cdot 1 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 2 \cdot 4 = 30$$

Let $P_2 = (Q_2, \Sigma, W, \mu_2, \pi_2, \eta_2)$ be a pwfa , where $Q_2 = \{q_3, q_4\}$, $\Sigma = \{a, b\}$,
$\mu_2 : Q_2 \times \Sigma \times Q_2 \to [0, \infty)$ is defined as follows:
$\mu_2(q_3, a, q_3) = 2$, $\mu_2(q_3, a, q_4) = 3$, $\mu_2(q_4, b, q_4) = 4$
and $\pi_2(q_3) = 2, \pi_2(q_4) = 1$; $\eta_2(q_3) = 2, \eta_2(q_4) = 3$.
The language accepted by P_2 is a weighted subset $L_2 : \Sigma^* \to [0, \infty)$ such that

$$L_2(x) = \begin{cases} w_1, & w_1 \geq 4 \quad if \quad x \in a^* \\ w_2, & w_2 \geq 18 \quad if \quad x \in a^* ab^* \\ w_3, & w_3 \geq 3 \quad if \quad x \in b^* \\ 0, & otherwise \end{cases}$$

The transition diagram is shown below:

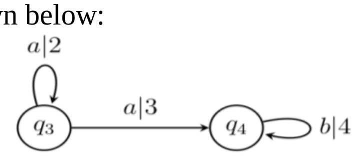

Figure 3.2

Consider $w = aa$ then $L_2(\mathrm{aa}) = 52$.

Let $\mathrm{P} = \mathrm{P}_1 \cap P_2 = (Q_1 \times Q_2, \Sigma, \mathrm{W}, \mu_1 \times \mu_2, \pi_1 \times \pi_2, \eta_1 \times \eta_2)$,

$Q = \{(q_1, q_3), (q_1, q_4), (q_2, q_3), (\mathrm{q}_2, \mathrm{q}_4)\}$ and $\mu_1 \times \mu_2 : (Q_1 \times Q_2) \times \Sigma \times (Q_1 \times Q_2) \to [0, \infty)$ is defined by

$(\mu_1 \times \mu_2)((\mathrm{q}_1, \mathrm{q}_3), \mathrm{a}, (\mathrm{q}_1, q_4)) = 3$, $\quad (\mu_1 \times \mu_2)((\mathrm{q}_1, \mathrm{q}_3), \mathrm{a}, (\mathrm{q}_2, q_4)) = 6$,

$(\mu_1 \times \mu_2)((\mathrm{q}_1, \mathrm{q}_3), \mathrm{a}, (\mathrm{q}_2, q_3)) = 4$, $\quad (\mu_1 \times \mu_2)((\mathrm{q}_1, \mathrm{q}_3), \mathrm{a}, (\mathrm{q}_1, q_3)) = 2$,

$(\mu_1 \times \mu_2)((\mathrm{q}_1, \mathrm{q}_4), \mathrm{b}, (\mathrm{q}_1, q_4)) = 8$.

$(\pi_1 \times \pi_2)(p_1, p_2) : Q_1 \times Q_2 \to [0, \infty)$ is defined by

$(\pi_1 \times \pi_2)(\mathrm{q}_1, \mathrm{q}_3) = 6$, $\quad (\pi_1 \times \pi_2)(\mathrm{q}_2, \mathrm{q}_3) = 2$,

$(\pi_1 \times \pi_2)(\mathrm{q}_1, \mathrm{q}_4) = 3$, $\quad (\pi_1 \times \pi_2)(\mathrm{q}_2, \mathrm{q}_4) = 1$.

$(\eta_1 \times \eta_2) : Q_1 \times Q_2 \to [0, \infty)$ is defined by

$(\eta_1 \times \eta_2)(\mathrm{q}_1, \mathrm{q}_3) = 4$, $\quad (\eta_1 \times \eta_2)(\mathrm{q}_2, \mathrm{q}_3) = 8$

$(\eta_1 \times \eta_2)(\mathrm{q}_1, \mathrm{q}_4) = 6$, $\quad (\eta_1 \times \eta_2)(\mathrm{q}_2, \mathrm{q}_4) = 12$.

The language accepted by P is a weighted subset $L : \Sigma^* \to [0, \infty)$ such that

$$L(x) = \begin{cases} w_1, & w_1 \geq 732 \quad if \quad x \in aa^* \\ w_2, & w_2 \geq 24 \quad if \quad x \in a^* \\ \mathrm{w}_3, & w_3 \geq 18 \quad if \quad x \in b^* \\ 0, & otherwise \end{cases}$$

$L(\mathrm{aa}) = 6 \cdot 2 \cdot 2 \cdot 4 + 6 \cdot 2 \cdot 3 \cdot 6 + 6 \cdot 2 \cdot 6 \cdot 12 + 6 \cdot 2 \cdot 4 \cdot 8 = 1560$

$L_1(\mathrm{aa}) \cdot L_2(\mathrm{aa}) = 30 \cdot 52 = 1560 = L(\mathrm{aa})$ .Thus $\quad L(x) = L_1(x) \cdot L_2(x)$

The transition diagram is shown below:

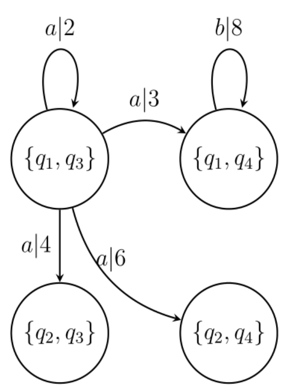

Figure 3.3 Example for an intersection of pwfa.

Theorem 3.6. Let A and B be a recognizable set over $\sum^*$ with weighted regular languages L_1 and L_2, which is accepted by pwfa's P_1 and P_2 respectively. Then the set AB is recognizable by a pwfa P such that $L = L_1 L_2$.

Proof: Let $P_1 = (Q_1, \Sigma, W, \mu_1, \pi_1, \eta_1)$ and $P_2 = (Q_2, \Sigma, W, \mu_2, \pi_2, \eta_2)$ be two pwfa's, $Q_1 \cap Q_2 = \varnothing$ with weighted regular languages be L_1 and L_2 respectively. i.e., $L_1 : \Sigma^* \to [0, \infty)$ such that

$L_1(x) > 0$ for all $x \in A$ $L_2 : \Sigma^* \to [0, \infty)$ such that $L_2(x) > 0$ $\forall$ $x \in B$.

Let $Q = Q_1 \cup Q_2$. Define pwfa $P = (Q, \Sigma, W, \mu, \pi, \eta)$ where $\mu : Q \times \Sigma \times Q \to [0, \infty)$ is defined as follows:

(i) $\forall\, p, q \in Q_1, a \in \Sigma, \mu(p, a, q) = \mu_1(p, a, q)$

(ii) $\forall\, p, q \in Q_2, a \in \Sigma, \mu(p, a, q) = \mu_2(p, a, q)$

(iii) $\forall\, p \in Q_1, q \in Q_2$ and if $\eta_1(p) > 0$, $r \in Q_2$ with $\pi_2(r) > 0$ and $\mu_2(r, a, q) > 0$ then

$$\mu(p, a, q) = \begin{cases} \eta_1(p) \cdot \pi_2(r) \cdot \mu_2(r, a, q) \\ 0, \quad otherwise \end{cases}$$

$\pi : Q \to [0, \infty)$ is defined by

$$\pi(p) = \begin{cases} \pi_1(p), & if \quad p \in Q_1, \\ 0, & otherwise \end{cases}$$

$\eta : Q \to [0, \infty)$ is defined by

$$\eta(p) = \begin{cases} \eta_2(p), & if \quad p \in Q_2, \\ 0, & otherwise \end{cases}$$

Let $L(z) > 0$. Suppose $L(z) = \sum_{p,q \in Q} \pi(p) \cdot \mu^*(p, z, q) \cdot \eta(q)$ (3)

Since Q is finite and z is of finite length, $L(z)$ is finite. Let $z = xy$ where $x = a_1 a_2 ... a_n \in A$, $y = b_1 b_2 ... b_m \in B$. Consider any term in $L(z)$. Let it be $w(xy)$ where $x \in A, y \in B$.

$$\begin{aligned} w(xy) &= \pi(p_0) \cdot \mu^*(p_0, xy, q_m) \cdot \eta(q_m) \\ &= \pi(p_0) \cdot \mu(p_0, a_1, p_1) \cdot \mu(p_1, a_2, p_2) \cdots \mu(p_{n-1}, a_n, p_n) \cdot \mu(p_n, b_1, q_1) \cdot \mu(q_1, b_2, q_2) \cdots \\ &\quad \mu(q_{m-1}, b_m, q_m) \cdot \eta(q_m) \end{aligned}$$

From the definition of P we have,

$$\begin{aligned} w(xy) &= \pi_1(p_0) \cdot \mu_1(p_0, a_1, p_1) \cdot \mu_1(p_1, a_2, p_2) \cdots \mu_1(p_{n-1}, a_n, p_n) \cdot \eta_1(p_n) \cdot \pi_2(q_0) \cdot \mu_1(q_0, b_1, q_1) \cdot \\ &\quad \mu_2(q_1, b_2, q_2) \cdots \mu_2(q_{m-1}, b_m, q_m) \cdot \eta_2(q_m) \end{aligned} \quad (4)$$

$= w_1(x)\cdot w_2(y)$. Thus $w(xy) = w_1(x)\cdot w_2(y)$. This is true for every term in (3). Then,

$$\sum_{p,q\in Q} \pi(p)\cdot \mu^*(p,z,q)\cdot \eta(q) = \left\{\sum_{p_0,p_n\in Q_1} \pi_1(p_0)\cdot \mu_1^*(p_0,x,p_n)\cdot \eta_1(p_n)\right\}\cdot\left\{\sum_{q_0,q_m\in Q_2} \pi_2(q_0)\cdot \mu_2^*(q_0,y,q_m)\cdot \eta_2(q_m)\right\}$$

$= L_1(x)\cdot L_2(y)$. Thus $L(xy) = L_1(x)\cdot L_2(y)$.

Let $x\in A, y\in B$. Suppose $L_1(x) > 0$, $L_2(y) > 0$, then

$$L_1(x) = \sum_{p_0,p_n\in Q_1} \pi_1(p_0)\cdot \mu_1^*(p_0,x,p_n)\cdot \eta_1(p_n) \tag{5}$$

Now, consider any term in $L_1(x)$. Let it be $w_1(x)$

$$w_1(x) = \pi_1(p_0)\cdot \mu_1^*(p_0,x,p_n)\cdot \eta_1(p_n)$$

$$= \pi_1(p_0)\cdot \mu_1(p_0,a_1,p_1)\cdot \mu_1(p_1,a_2,p_2)\cdots \mu_1(p_{n-1},a_n,p_n)\cdot \eta_1(p_n)$$

where $\pi_1(p_0) > 0$, $\eta_1(p_n) > 0$, $x = a_1a_2...a_n \in A$, $p_0, p_1....p_n \in Q_1$

Similarly if $L_2(y) = \sum_{q_0,q_m\in Q_2} \pi_2(q_0)\cdot \mu_2^*(q_0,y,q_m)\cdot \eta_2(p_m)$ (6)

$$w_2(y) = \pi_2(q_0)\cdot \mu_2^*(q_0,y,q_m)\cdot \eta_2(q_m)$$

$$= \pi_2(q_0)\cdot \mu_2(q_0,b_1,q_1)\cdot \mu_2(q_1,b_2,q_2)\cdots \mu_2(q_{m-1},b_m,q_m)\cdot \eta_2(q_m)$$

where $\pi_2(q_0) > 0$, $\eta_2(q_m) > 0$, $x = b_1b_2...b_m \in B$, $q_0, q_1....q_m \in Q_2$

$$w_1(x)\cdot w_2(y) = \{\pi_1(p_0)\cdot \mu_1(p_0,a_1,p_1)\cdot \mu_1(p_1,a_2,p_2)\cdots \mu_1(p_{n-1},a_n,p_n)\cdot \eta_1(p_n)\}\cdot$$
$$\{\pi_2(q_0)\cdot \mu_2(q_0,b_1,q_1)\cdot \mu_2(q_1,b_2,q_2)\cdots \mu_2(q_{m-1},b_m,q_m)\cdot \eta_2(q_m)\}$$

From the definition of P,

$$w_1(x)\cdot w_2(y) = \pi(p_0)\cdot \mu(p_0,a_1,p_1)\cdot \mu(p_1,a_2,p_2)\cdots \mu(p_{n-1},a_n,p_n)\cdot$$
$$\mu(p_n,b_1,q_1)\cdot \mu(q_1,b_2,q_2)\cdots \mu(q_{m-1},b_m,q_m)\cdot \eta(q_m)$$

$= w(z)$. Thus $w_1(x)\cdot w_2(y) = w(z)$. Then,

$$\left\{\sum_{p_0,p_n\in Q_1} \pi_1(p_0)\cdot \mu_1^*(p_0,x,p_n)\cdot \eta_1(p_n)\right\}\cdot\left\{\sum_{q_0,q_m\in Q_2} \pi_2(q_0)\cdot \mu_2^*(q_0,y,q_m)\cdot \eta_2(p_m)\right\}$$

$= \sum_{p,q\in Q} \pi(p)\cdot \mu_1^*(p,xy,q)\cdot \eta(q)$. Thus $L_1(x)\cdot L_2(y) = L(x\,y)$. (by (3),(5),(6)

Example 3.7. Let $P_1 = (Q_1,\Sigma,W,\mu_1,\pi_1,\eta_1)$ be a pwfa where, $Q_1 = \{q_1,q_2,q_3\}$, $\Sigma = \{a,b\}$, $\mu_1 : Q_1\times\Sigma\times Q_1 \to [0,\infty)$ is defined as follows:

$\mu_1(q_1,a,q_1) = 2$, $\mu_1(q_2,b,q_2) = 4$, $\mu_1(q_1,b,q_3) = 1$,

$\mu_1(q_2,a,q_3) = 3$, $\mu_1(q_1,a,q_2) = 2$.

and $\pi_1(q_1)=1, \pi_1(q_2)=1.5;\quad \eta_1(q_3)=2$.

The transition diagram is shown below:

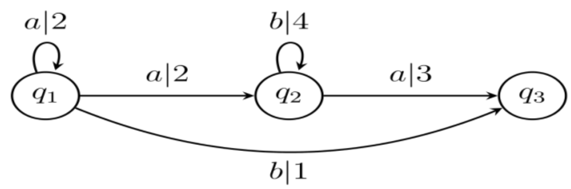

Figure 3.4

The language accepted by P_1 is a weighted subset $L_1 : \Sigma^* \to [0,\infty)$ such that

$$L_1(x)=\begin{cases} w_1, & w_1 \geq 12 \quad if \quad x \in a^* ab^* a \\ w_2, & w_2 \geq 2 \quad if \quad x \in a^* b \\ w_3, & w_3 \geq 9 \quad if \quad x \in b^* a \\ 0, & otherwise \end{cases}$$

Let $P_2=(Q_2,\Sigma,W,\mu_2,\pi_2,\eta_2)$ be a pwfa , where $Q_2=\{q_4,q_5,q_6\}$, $\Sigma=\{a,b\}$, $\mu_2 : Q_2 \times \Sigma \times Q_2 \to [0,\infty)$ is defined as follows:

$\mu_2(q_4,a,q_5)=3$, $\quad \mu_2(q_4,b,q_5)=2$,

$\mu_2(q_5,b,q_6)=1$, $\quad \mu_2(q_6,a,q_6)=2$.

and $\pi_2(q_4)=2, \pi_2(q_5)=1.5;\quad \eta_2(q_6)=3$.

The transition diagram is shown below:

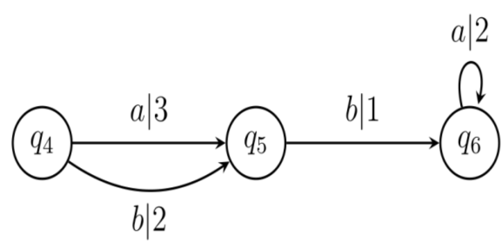

Figure 3.5

The language accepted by P_2 is a weighted subset $L_2 : \Sigma^* \to [0,\infty)$ such that

$$L_2(x)=\begin{cases} w_1, & w_1 \geq 18 \quad if \quad x \in aba^* \\ w_2, & w_2 \geq 12 \quad if \quad x \in bba^* \\ w_3, & w_3 \geq 4.5 \quad if \quad x \in ba^* \\ 0, & otherwise \end{cases}$$

Let $P=(Q,\sum,W,\mu,\pi,\eta)$ be a pwfa, where $Q=Q_1\cup Q_2=\{q_1,q_2,q_3,q_4,q_5,q_6\}$, $\sum=\{a,b\}$, $\mu:Q\times\sum\times Q\rightarrow[0,\infty)$ is defined as follows:

$\mu(q_1,a,q_1)=2,\qquad \mu(q_2,b,q_2)=4,\qquad \mu(q_1,b,q_3)=1,$

$\mu(q_2,a,q_3)=3,\qquad \mu(q_1,a,q_2)=2,\qquad \mu(q_4,a,q_5)=3,$

$\mu(q_4,b,q_5)=2,\qquad \mu(q_5,b,q_6)=1,\qquad \mu(q_6,a,q_6)=2$

$\mu(q_3,a,q_5)=12,\qquad \mu(q_3,b,q_5)=8,\qquad \mu(q_3,b,q_6)=3$

and $\pi(q_1)=1;\qquad \eta(q_6)=3$.

The transition diagram is shown below:

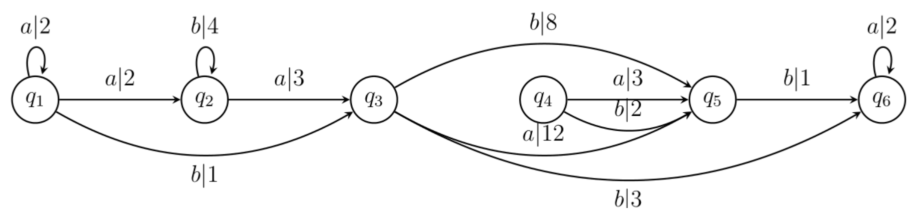

Figure 3.6

The language accepted by P is a weighted subset $L:\sum^*\rightarrow[0,\infty)$ such that

$$L(x)=\begin{cases} w_1, & w_1\geq 216 \quad if \quad x\in a^*ab^*abba^* \\ w_2, & w_2\geq 36 \quad if \quad x\in a^*baba^* \\ w_3, & w_3\geq 54 \quad if \quad x\in a^*ab^*aba^* \\ w_4, & w_4\geq 144 \quad if \quad x\in a^*ab^*abba^* \\ w_5, & w_5\geq 24 \quad if \quad x\in a^*bbba^* \\ w_6, & w_6\geq 9 \quad if \quad x\in a^*bba^* \\ w_7, & w_7\geq 108 \quad if \quad x\in b^*abba^* \\ w_8, & w_8\geq 40.5 \quad if \quad x\in b^*aba^* \\ w_9, & w_9\geq 162 \quad if \quad x\in b^*aaba^* \\ 0, & otherwise \end{cases}$$

Consider $z=baba$ then we have $L_1(\text{ba})=36,\qquad L_2(\text{ba})=9$

$L(\text{baba})=324$, $L_1(\text{ba})\cdot L_2(\text{ba})=36\cdot 9=324$. Thus $L(\text{baba})=L_1(\text{ba})\cdot L_2(\text{ba})$

Theorem 3.8. Let $A\subseteq\sum^*$ be recognizable set with weighted regular languages L_1, which is accepted by pwfa P_1. Then A^* is recognizable with weighted regular languages L, accepted by a pwfa P such that $L=L^*_1$.

Proof: Let $P_1 = (Q_1, \Sigma, W, \mu_1, \pi_1, \eta_1)$ be a pwfa with L_1 where $Q = Q_1$, $\mu : Q \times \Sigma \times Q \to [0, \infty)$ is defined as follows :

(i)

$$\mu(p,a,q) = \mu_1(p,a,q) \quad \forall p,q \in Q, \quad a \in \Sigma$$

(ii) For $p,q \in Q$, if $r \in Q_1$, $\pi_1(q) > 0, \quad \eta_1(r) > 0, \quad \mu_1(p,a,r) > 0$ then include

$$\mu(p,a,q) = \mu_1(p,a,r) \cdot \eta_1(r) \cdot \pi_1(q)$$

$\pi_1 : Q_1 \to [0,\infty)$ defined by $\pi(q) = \pi_1(q) \quad \forall q \in Q_1$

$\eta_1 : Q_1 \to [0,\infty)$ defined by $\eta(q) = \eta_1(q) \; \forall q \in Q_1$

Let $z \in A^*, \quad L(z) > 0$ then, $L(z) = \sum_{p,q \in Q} \pi(p) \cdot \mu^*(p,z,q) \cdot \eta(q)$ (7)

Since Q is finite and $z = x_1 x_2 \cdots\cdots x_m$ is of finite length, we consider any term in $L(z)$, let it be $w(z)$. Then,

$$w(x_1 x_2 \cdots\cdots x_m) \quad = \pi(p) \cdot \mu^*(p_1, x_1 x_2 \cdots x_m, p_{m+1}) \cdot \eta(p_{m+1})$$

$$\begin{aligned} w(z) = \pi\ (p) \cdot \mu(p_1, a_{11}, p_{11}) \cdot \mu(p_{11}, a_{12}, p_{12}) \cdots\cdots \mu(p_{1n_1-1}, a_{1n_1}, p_2) \cdot \mu(p_2, a_{21}, p_{21}) \cdot \\ \mu(p_{21}, a_{22}, p_{22}) \cdots \mu(p_{2n_2-1}, a_{2n_2}, p_3) \cdots \mu(p_m, a_m, p_{m1}) \cdots \\ \mu(p_{mn_m-1}, a_{mn_m}, p_{m+1}) \cdot \eta(p_{m+1}) \end{aligned} \quad (8)$$

where $p_i, p_{ij}, p_{m+1} \in Q, \quad i = 1,2,3.....m, \quad j = 1,2,3.....(n_i - 1), \quad x_i = a_{i1} a_{i2} ....a_{in_i} \in A$ and

$z = x_1 x_{2.} ......x_m$

From the definition of P, we have

$$\begin{aligned} w(z) = \pi_1(p_1).\mu_1(p_1, a_{11}, p_{11}) \cdot \mu_1(p_{11}, a_{12}, p_{12}) \cdots\cdots \mu_1(p_{1n_1-1}, a_{1n_1}, p_{1n_1}) \cdot \eta_1(p_{1n_1}) \cdot \pi_1(p_2) \cdot \\ \mu_1(p_2, a_{21}, p_{21}) \cdots \mu_1(p_{2n_2-1}, a_{2n_2}, p_{2n_2}) \cdot \eta_1(p_{2n_2}) \cdots\cdots \pi_1(p_m) \cdot \mu_1(p_m, a_{m1}, p_{m1}) \cdots \\ \cdot \mu_1(p_{mn_m-1}, a_{mn_m}, p_{m+1}) \cdot \eta_1(p_{m+1}) \end{aligned}$$

$$\begin{aligned} w(x_1 x_2 \cdots\cdots x_m) = \left(\pi_1(p_1) \cdot \mu_1^*(p_1, x_1, p_{1n_1}) \cdot \eta_1(p_{1n_1})\right) \cdot \left(\pi_1(p_2) \cdot \mu_1^*(p_2, x_2, p_{2n_2}) \cdot \eta_1(p_{2n_2})\right) \cdots \\ \left(\pi_1(p_m) \cdot \mu_1^*(p_m, x_m, p_{m+1}) \cdot \eta_1(p_{m+1})\right) \\ = w_1(x_1) \cdot w_1(x_2) \cdot w_1(x_3) \cdots w_1(x_m) \end{aligned}$$

This is true for every term in (7)

$$\sum_{p,q\in Q}\pi(p)\cdot\mu^*(p,z,q)\cdot\eta(q)=\sum_{p_1,p_{1n_1}\in Q_1}\pi_1(p_1)\cdot\mu_1^*(p_1,x_1,p_{1n_1})\cdot\eta_1(p_{1n_1})\cdot\sum_{p_2,p_{2n_2}\in Q_1}\pi_1(p_2)\cdot\mu_1^*(p_2,x_2,p_{2n_2})\cdot\eta_1(p_{2n_2})\cdot$$
$$\cdots\sum_{p_m,p_{m+1}\in Q_1}\pi_1(p_m)\cdot\mu_1^*(p_m,x_m,p_{m+1})\cdot\eta_1(p_{m+1})$$
$$i.e., L(z)=L_1(x_1)\cdot L_1(x_2)\cdots L_1(x_m)$$

Let $x_i\in A,$ if $L_1(x_i)>0,\quad i=1,2,........m$

then $L_1(x_i)=\sum_{p,q\in Q_1}\pi_1(p)\cdot\mu_1^*(p,x_i,q)\cdot\eta_1(q).$ Since Q_1 is finite and $z=x_1x_2\cdots\cdots x_m$ is of finite length .Therefore $L_1(x_i)$ is of finite length. Now consider any term in $L_1(x_i)$, let it be $w_1(x_i)$, $i=1,2,3.....m,$

$$w_1(x_1)=\pi_1(p_1).\mu_1(p_1,a_{11},p_{11})\cdot\mu_1(p_{11},a_{12},p_{12})\cdots\cdots\mu_1(p_{1n_1-1},a_{1n_1},p_{1n_1})\cdot\eta_1(p_{1n_1})$$
$$w_1(x_2)=\pi_1(p_2)\cdot\mu_1(p_2,a_{21},p_{21})\cdot\mu_1(p_{21},a_{22},p_{22})\cdots\mu_1(p_{2n_2-1},a_{2n_2},p_{2n_2})\cdot\eta_1(p_{2n_2})$$
$$\vdots$$
$$w_1(x_m)=\pi_1(p_m)\cdot\mu_1(p_m,a_{m1},p_{m1})\cdot\mu_1(p_{m1},a_{m2},p_{m2})\cdots\mu_1(p_{mn_m-1},a_{mn_m},p_{m+1})\cdot\eta_1(p_{m+1})$$
$$w_1(x_1)\cdot w_1(x_2)\cdot w_1(x_3)\cdots w_1(x_m)=\pi_1(p_1).\mu_1(p_1,a_{11},p_{11})\cdot\mu_1(p_{11},a_{12},p_{12})\cdots\mu_1(p_{1n_1-1},a_{1n_1},p_{1n_1})\cdot\eta_1(p_{1n_1})$$
$$\cdot\pi_1(p_2)\cdot\mu_1(p_2,a_{21},p_{21})\cdots\mu_1(p_{2n_2-1},a_{2n_2},p_{2n_2})\cdot\eta_1(p_{2n_2}).$$
$$...\pi_1(p_m)\cdot\mu_1(p_m,a_{m1},p_{m1})\cdots\cdots\mu_1(p_{mn_m-1},a_{mn_m},p_{m+1})\cdot\eta_1(p_{m+1})$$

From the definition of P, we have

$w_1(x_1)\cdot w_1(x_2)\cdot w_1(x_3)\cdots w_1(x_m)=w(x_1x_2\cdots\cdots x_m)$ .Then

$$\left\{\sum_{p_1,p_{1n_1}\in Q_1}\pi_1(p_1)\cdot\mu_1^*(p_1,x_1,p_{1n_1})\cdot\eta_1(p_{1n_1})\right\}\cdot\left\{\sum_{p_2,p_{2n_2}\in Q_1}\pi_1(p_2)\cdot\mu_1^*(p_2,x_2,p_{2n_2})\cdot\eta_1(p_{2n_2})\right\}$$
$$\cdots\left\{\sum_{p_m,p_{m+1}\in Q_1}\pi_1(p_m)\cdot\mu_1^*(p_m,x_m,p_{m+1})\cdot\eta_1(p_{m+1})\right\}=\sum_{p_1,p_{m+1}\in Q}\pi(p_1)\cdot\mu^*(p_1,z,p_{m+1})\cdot\eta(p_{m+1})$$

i.e., $L_1(x_1)\cdot L_1(x_2)\cdots L_1(x_m)=L(z)$

4. CONCLUSION

In this paper, the results of weighted automata are extended for plus weighted finite automata (pwfa). We have made an attempt to study on some properties of pwfa. We have made a humble beginning in this direction, however, many concepts are yet to be changed into weighted automata in the context of pwfa.

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