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A Characterization of the $L^2(\mathbb{R})$ Space

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ABSTRACT

Let (n_k) be a lacunary sequence with no non-trivial common divisor. Define the square function

$$Sf(x)=\left(\sum_{k=1}^{\infty}\left|\frac{1}{n_{k+1}}\int_0^{n_{k+1}} f(x-t)dt-\frac{1}{n_k}\int_0^{n_k} f(x-t)dt\right|^2\right)^{1/2}.$$

We prove that Sf characterizes the $L^2(\mathbb{R})$ space.

Mathematics Subject Classification: 26D07, 42B20.

Keywords: Square Function, $L^2(\mathbb{R})$ Space.

INTRODUCTION

Let (n_k) be an increasing sequence of positive integers. Define the square function

$$Sf(x)=\left(\sum_{k=1}^{\infty}\left|\frac{1}{n_{k+1}}\int_0^{n_{k+1}} f(x-t)dt-\frac{1}{n_k}\int_0^{n_k} f(x-t)dt\right|^2\right)^{1/2}.$$

In this article we will show that Sf characterizes the $L^2(\mathbb{R})$ space when the sequence (n_k) is lacunary with no non-trivial common divisor.

We will first show that Sf satisfies a strong L^2 inequality without the lacunarity condition on the sequence (n_k).

Theorem 1. Let (n_k) be an increasing sequence of positive integers then we have

$$\|Sf\|_2 \leq 25\|f\|_2$$

for all $f \in L^2(\mathbb{R})$.

Proof. Let $\phi_n(x) = \frac{1}{n}\chi_{[0,n]}(x)$ where χ_E denotes the chararacteristic function of the set E.
Then obviously we have

$$Sf(x) = \left(\sum_{k=1}^{\infty} \left| \phi_{n_{k+1}} * f(x) - \phi_{n_k} * f(x) \right|^2 \right)^{1/2}.$$

On the other hand, it is easy to check that

$$\left| \hat{\phi}_{n_{k+1}}(x) - \hat{\phi}_{n_k}(x) \right| = \left| \frac{\sin(n_{k+1}x)}{n_{k+1}x} - \frac{\sin(n_k x)}{n_k x} \right|$$

$$\leq \left| \frac{\gamma^{n_{k+1}}}{n_{k+1}(\gamma - 1)} - \frac{\gamma^{n_k}}{n_k(\gamma - 1)} \right|$$

where $\gamma = e^{it}$ with $-\pi \leq t \leq \pi$.
But we then see by using the estimate in the proof of Theorem 1.2 in R. L. Jones *et al.*,[2] that

$$\sum_{k=1}^{\infty} \left| \hat{\phi}_{n_{k+1}}(x) - \hat{\phi}_{n_k}(x) \right|^2 \leq 25^2.$$

Thus we have

$$\|Sf\|_2^2 = \int \sum_{k=1}^{\infty} \left| \phi_{n_{k+1}} * f(x) - \phi_{n_k} * f(x) \right|^2 dx$$

$$= \sum_{k=1}^{\infty} \int \left| \phi_{n_{k+1}} * f(x) - \phi_{n_k} * f(x) \right|^2 dx$$

$$= \sum_{k=1}^{\infty} \int \left| \phi_{n_{k+1}} * f(x) - \phi_{n_k} * f(x) \right|^2 dx$$

(by Plancherel Theorem)

$$= \int \sum_{k=1}^{\infty} \left| \phi_{n_{k+1}} * f(x) - \phi_{n_k} * f(x) \right|^2 dx$$

$$= \int \sum_{k=1}^{\infty} \left| \hat{\phi}_{n_{k+1}}(x) - \hat{\phi}_k(x) \right|^2 \left| \hat{f}(x) \right|^2 dx$$

$$\leq 25^2 \int \left| \hat{f}(x) \right|^2 dx$$

$$= 25^2 \int |f(x)|^2 \, dx \quad \text{(by Plancherel Theorem)}$$

and thus our proof is complete.

Our next goal is to prove that the square function Sf satisfies a reverse L^2 norm inequality when the sequence (n_k) is lacunary with no non-trivial common divisor.

Recall that an incrasing sequence of positive integers is called lacunary if there exists a contsant $\beta > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq \beta$$

for all $k = 1, 2, 3, \ldots$.

The proof of the following lemma can be found in R. L. Jones and J. Rosenblatt[3]:

Lemma 1. There is a constant δ such that for any a and b with $0 < a < b$,

$$\inf_{\frac{1}{b} \leq t \leq \frac{1}{a}} \left| \frac{\sin(at)}{at} - \frac{\sin(bt)}{bt} \right| \geq \delta \left(1 - \frac{a}{b} \right).$$

Lemma 2. Let n_k be a lacunary sequence with no non-trivial common divisor. Then there exists a constant $C > 0$ such that

$$C \leq \sum_{k=1}^{\infty} \left| \phi_{n_{k+1}}(x) - \phi_{n_k}(x) \right|^2$$

where

$$\phi_n(x) = \frac{1}{n} \chi_{[0,n]}(x).$$

Proof. The argument of the proof of Theorem 1 in R. L. Jones and J. Rosenblatt[3] can be used to prove this result since

$$\left| \phi_{n_{k+1}} \quad x \; - \phi_{n_k} \quad x \; \right| = \left| \frac{\sin(n_{k+1}x)}{n_{k+1}x} - \frac{\sin(n_k x)}{n_k x} \right|$$

$$\geq \delta \left(1 - \frac{n_k}{n_{k+1}} \right)$$

by Lemma 1. Thus, when $\frac{1}{n_k} \geq |x| \geq \frac{1}{n_{k+1}}$ we have

$$\left| \frac{\sin(n_{k+1}x)}{n_{k+1}x} - \frac{\sin(n_k x)}{n_k x} \right| \geq \delta \left(1 - \frac{1}{\beta} \right) > 0.$$

Hence we have

$$\sum_{k=1}^{\infty}\left|\phi_{n_{k+1}}(x)-\phi_{n_k}(x)\right|^2=\sum_{k=1}^{\infty}\left|\frac{\sin(n_{k+1}x)}{n_{k+1}x}-\frac{\sin(n_k x)}{n_k x}\right|^2$$
$$\geq \delta^2\left(1-\frac{1}{\beta}\right)^2.$$

Our next result is the following:

Theorem 2. Let (n_k) be a lacunary sequence with no non-trivial common divisor. Then there exists a constant $C>0$ such that

$$\|f\|_2 \leq C\|Sf\|_2$$

for all $f \in L^2(\mathbb{R})$.

Proof. We have

$$\|f\|_2 = \left(\int |f(x)|^2\,dx\right)^{1/2}$$
$$=\left(\int |f(x)|^2\,dx\right)^{1/2} \quad \text{(by Plancherel theorem)}$$
$$\leq C\left(\int \sum_{k=1}^{\infty}|Q_k(x)|^2\,|f(x)|^2\,dx\right)^{1/2}$$
$$=C\left(\int \sum_{k=1}^{\infty}|Q_k(x)f(x)|^2\,dx\right)^{1/2}$$
$$=C\left(\int \sum_{k=1}^{\infty}|Q_k * f(x)|^2\,dx\right)^{1/2} \quad \text{(by Lemma 2)}$$
$$=C\left(\sum_{k=1}^{\infty}\int |Q_k * f(x)|^2\right)^{1/2}$$
$$=C\left(\sum_{k=1}^{\infty}\int |Q_k * f(x)|^2\right)^{1/2} \quad \text{(by Plancherel theorem)}$$
$$=C\left(\int \sum_{k=1}^{\infty}|Q_k * f(x)|^2\,dx\right)^{1/2}$$

$$= C\left(\int \left| \left(\sum_{k=1}^{\infty} |Q_k * f(x)|^2 \right)^{1/2} \right|^2 dx \right)^{1/2}$$
$$= C\|Sf\|_2$$

where C is a positive constant.

Corollary 3. The square function Sf characterizes $L^2(\mathbb{R})$ when the sequence n_k be a lacunary sequence with no non-trivial common divisor.

Proof. This follows from Theorem 1 and Theorem 2.

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