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Logarithmic Summability in Ergodic Theory

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ABSTRACT

Given an ergodic, measure preserving dynamical system (X, β, μ, τ) we prove under an additional assumption that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{f(\tau^k x)}{k} = \int f d\mu \quad \text{a.e.}$$

for all $f \in L^1(X)$. Another result about the behavior of the same sequence of operators has also been found.

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INTRODUCTION

Let (X, β, μ, τ) be an ergodic, measure preserving dynamical system. We will use the following well known theorem to prove our result.

Theorem 1 (Pointwise Ergodic Theorem). If $f \in L^1(X)$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\tau^k x) = \int f d\mu \quad \text{a.e.}$$

The next theorem is our main result.

Theorem 2. Let $f \in L^1(X)$ and $x \in X$. Then we have

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{f(\tau^k x)}{k}=\int f d\mu \text{ a.e.}$$

Suppose also that there are positive constants C_1 and C_2 such that for all $n \geq 1$,

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\tau^k x)-\int f d\mu\right| < C_1\frac{1}{n^{C_2}} .$$

Then there is a constant C which depends on C_1 and C_2 such that for $n > 1$,

$$\left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{f(\tau^k x)}{k}-\int f d\mu\right| \leq \frac{C}{\log n} .$$

Proof. We need to observe first that $0 \leq f(\tau^k x) \leq B$ for some constant $B < \infty$, for all k. All we know if the fact that $f \in L^1(X)$.

Let $f \in L^1(X)$ and choose an $x \in X$ such that pointwise ergodic theorem is satisfied. i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(\tau^k x)=\int f d\mu .$$

Then let

$$L=\int f d\mu$$

and define the sequence (a_n) and (b_n) as

$$a_n=\frac{1}{n}\sum_{k=1}^{n} f(\tau^k x)$$

and

$$b_n=\frac{1}{S_n}\sum_{k=1}^{n}\frac{f(\tau^k x)}{k},$$

where

$$S_n=\sum_{k=1}^{n}\frac{1}{k} .$$

Now define a transformation M by

$$M_{ij}=\begin{cases}\dfrac{1}{(j+1)S_i}, & if \quad i<j,\\[2mm] \dfrac{1}{S_i}, & if \quad i=j,\\[2mm] 0, & otherwise.\end{cases}$$

It is easy to see that the row sums of M are all 1 and for fixed j,

$$\lim_{i\to\infty} M_{ij} = 0.$$

It is also easy to check that $Ma = b$. Since the row sums of M are all 1, we have

$$b_n - L = M_{n1}(a_1 - L) + M_{n2}(a_2 - L) + \ldots + M_{nn}(a_n - L).$$

Suppose $\varepsilon > 0$ is given. Find an integer N_0 such that for all $k > N_0$,

$$|a_k - L| < \frac{\varepsilon}{2}.$$

Next choose N_1 big enough that for all $n > N_1$ and all $k \le N_0$, $M_{nk} \le \varepsilon / 4N_0 B$. Then for all $n > \max(N_0, N_1)$,

$$\begin{aligned} |b_n - L| &\le \sum_{k=1}^{N_0} M_{nk} |a_k - L| + \sum_{k=N_0+1}^{n} M_{nk} |a_k - L| \\ &\le \frac{\varepsilon}{4N_0 B} N_0 2B + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

We thus have

$$\lim_{n\to\infty} \frac{1}{S_n} \sum_{k=1}^{n} \frac{f(\tau^k x)}{k} = L.$$

Then by using the fact that $S_n \sim \log n + O(1)$ we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{f(\tau^k x)}{k} = L.$$

But pointwise ergodic theorem is true for a.e. x and therefore we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{f(\tau^k x)}{k} = \int f d\mu \text{ a.e.}$$

and this completes the proof of the first part of our theorem.

To prove the second part of our theorem first we use the equation

$$b_n - L = M_{n1}(a_1 - L) + M_{n2}(a_2 - L) + \ldots + M_{nn}(a_n - L)$$

and the definition of M to find

$$|b_n - L| \le \sum_{k=1}^{n} M_{nk} |a_k - L|$$

$$\leq \frac{1}{S_n}\left(\frac{C_1}{n^{C_2}}+\sum_{k=1}^{n-1}\frac{C_1}{(k+1)k^{C_2}}\right)$$
$$\leq \frac{C_1}{S_n}\left(1+\sum_{k=1}^{n-1}\frac{1}{k^{1+C_2}}\right).$$

Since C_2 is positive, the sum

$$\sum_{k=1}^{\infty}\frac{1}{k^{1+C_2}}$$

converges. Also, since $S_n \sim logn + O(1)$, we can find a constant C such that

$$|b_n - L| \leq \frac{C}{\log n},$$

and this completes the proof of the second part of our theorem as desired.

Remark 1. It is clear that our argument can also be used to find the same resut for other sequences of operators, i.e., let (T_k) be a sequence of operators defined on $L^p(X)$, for $1 \leq p \leq \infty$. Suppose that there is an $L_f < \infty$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}T_k f(x) = L_f \text{ a.e.}$$

for all $f \in L^p(X)$, $1 \leq p \leq \infty$. Also, assume that given $x \in X$ there is a constant B such that $0 \leq T_k f(x) \leq B$ for all k. Then we have

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{T_k f(x)}{k} = L_f \text{ a.e.}$$

As in our theorem if we assume also that there are positive constants C_1 and C_2 such that for all $n \geq 1$,

$$\left|\frac{1}{n}\sum_{k=1}^{n}T_k f(x) - L_f\right| < C_1\frac{1}{n^{C_2}}.$$

Then there is a constant C which depends on C_1 and C_2 such that for $n > 1$

$$\left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{T_k f(x)}{k} - L_f\right| \leq \frac{C}{\log n}.$$