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A Generalization of Calderón Transfer Principle

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ABSTRACT

Let T be an operator defined on the space of locally integrable functions on the real line with the following properties: the values of T are continuous functions on the real line, T is sublinear and commutes with the translations, and T is semilocal. Assume that X is a measure space which is totally σ-finite and U^t is a one-parameter group of measure-preserving transformations of X. Let us also assume that for every measurable function f on X the function $f(U^t x)$ is measurable in the product of X with the real line. Given a function f on X let $F(t,x)=f(U^t x)$. If f is the sum of two functions which are bounded and integrable, respectively, then $F(t,x)$ is locally integrable function of t for almost all x and thus $G(t,x)=T(F(t,x))$ is a well-defined continuous function of t for almost all x. Define $T^{\#}f=G(0,x)$. Let (S_n) and (K_n) be the sequences of operators with the above properties and define $Sf=\sup_n |S_n f|$ and $Kf=\sup_n |K_n f|$. Let $S^{\#}f=\sup_n |S_n^{\#} f|$ and $K^{\#}f=\sup_n |K_n^{\#} f|$. We prove that if there exists a constant $C>0$ such that $\|Sf\|_p \le C\|Kf\|_p$ for all $f\in L^p(\mathbb{R})$, $1\le p\le\infty$, then $\|S^{\#}f\|_p \le C\|K^{\#}f\|_p$ for all $f\in L^p(X)$, $1\le p\le\infty$.

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INTRODUCTION

We will assume that X is a measure space which is totally σ-finite and U^t is a one-parameter group of measure-preserving transformations of X. We will also assume that for every measureable function f on $g(x)=G(0,x)$ the function $f(U^t x)$ is measurable in the product of X with the real line. T will denote an operator defined on the space of locally integrable functions on the real line with the following properties: the values of T are continuous functions on the real line, T is sublinear and commutes with translations, and T is semilocal in the sense that there exists a positive number ε such that the support of Tf is always contained in an ε-neighborhood of the support of f. We will associate an operator $T^{\#}$ on the functions on X with such an operator T as follows:
Given a function f on X let

$$F(t,x)=f(U^t x).$$

If f is the sum of two functions which are bounded and integrable, respectively, then $F(t,x)$ is a locally integrable function of t for almost all x and therefore

$$G(t,x)=T(F(t,x))$$

İs a well-defined continuous function of t for almost all x. Thus $g(x)=G(0,x)$ has a meaning and we define

$$T^{\#}f=g(x).$$

Let now (S_n) be a sequence of operators as above and define

$$Sf=\sup_n |S_n f|$$

and

$$S^{\#}f=\sup_n |S_n^{\#} f|.$$

The proof of the following theorem can be found in A. P. Calderón[1].

Theorem 1. If there is a constant $C>0$ such that

$$\|Sf\|_p \le C\|f\|_p$$

for all $f\in L^p(\mathbb{R})$, $1\le p\le\infty$, then we have

$$\|S^{\#}f\|_p \le C\|f\|_p$$

for all $f\in L^p(X)$, $1\le p\le\infty$.

Now let (K_n) be a sequence of operators as above and define

$$Kf = \sup_n |K_n f|$$

and

$$K^{\#} f = \sup_n |K_n^{\#} f|.$$

Then we have the following result.

Theorem 2. If there is a constant $C > 0$ such that

$$\|Sf\|_p \leq C\|Kf\|_p$$

for all $f \in L^p(\mathbb{R}),\ 1 \leq p \leq \infty$, then we have

$$\|S^{\#} f\|_p \leq C\|K^{\#} f\|_p$$

for all $f \in L^p(X),\ 1 \leq p \leq \infty$.

Proof: We adapt the argument of A. P. Calderón[1] to our setting to prove our result. We may assume without loss of generality that the sequence (S_n) and (K_n) are finite, for if the theorem is established in this case, the general case follows by a passage to limit. Under this assumption the operators S and K have the same properties as the operators T above. As we did above, let

$$G(t,x) = S(F(t,x))$$

and

$$H(t,x) = K(F(t,x)).$$

We then have

$$S^{\#} f = G(0,x)$$

and

$$K^{\#} f = H(0,x).$$

We note that

$$F(t,U^s x) = F(t+s,x),$$

Which means that for any two given values t_1, t_2 of t, $F(t_1,x)$ and $F(t_2,x)$ are equimeasurable functions of x. On the other hand, due to translation invariance of S, the function $G(t,x)$ has the same property. In fact we have

$$G(t,U^s x) = S(F(t,U^s x)) = S(F(t+s,x)) = G(t+s,x).$$

Let now $F_a(t,x) = F(t,x)$ if $|t| < a$, $F_a(t,x) = 0$ otherwise, and let

$$G_a(t,x) = S(F_a(t,x)).$$

Since S is positive (i.e, its values are nonnegative functions) and sublinear, we have

$$G(t,x) = S(F) = S(F_{a+\varepsilon} + (F - F_{a+\varepsilon}))$$
$$\leq S(F_{a+\varepsilon}) + S(F - F_{a+\varepsilon})$$

and since $F - F_{a+\varepsilon}$ has supported in $|t| > a+\varepsilon$, and S is semilocal, the last term on the right vanishes for $|t| \leq a$ for ε sufficiently large, independently of a. Thus we have $G \leq G_{a+\varepsilon}$ for $|t| \leq a$. Suppose that S is of strong type (p,p). Then since $G(0,x)$ and $G(t,x)$ are equimeasurable functions of x, we have

$$2\int_X G(0,x)^p\,dx = \frac{1}{a}\int_{|t|<a} dt \int_X G(t,x)^p\,dx$$
$$\leq \frac{1}{a}\int_{|t|<a} dt \int_X G_{a+\varepsilon}(t,x)^p\,dx$$
$$= \frac{1}{a}\int_X dx \int_{|t|<a} G_{a+\varepsilon}(t,x)^p\,dt$$

and since $SF_{a+\varepsilon} = G_{a+\varepsilon}$ and $KF_{a+\varepsilon} = H_{a+\varepsilon}$ we have

$$\int_{|t|<a} G_{a+\varepsilon}(t,x)^p\,dt \leq C^p \int |\,H_{a+\varepsilon}(t,x)^p\,dt\,.$$

Thus we find

$$2\int_X G(0,x)^p\,dx \leq \frac{1}{a}C^p \int_X dx \int |\,H_{a+\varepsilon}(t,x)|^p\;dt$$

and again, since $H(0,x)$ and $H(t,x)$ are equimeasurable, the last integral is equal to

$$2(a+\varepsilon)\int_X |\,H(0,x)|^p\;dx$$

and

$$\int_X G(0,x)^p\,dx \leq \frac{1}{a}(a+\varepsilon)C^p \int_X |\,H(0,x)|^p\;dx\,.$$

Now let a tend to infinity, we find the desired result when $p < \infty$. When $p = \infty$, the assertion of the theorem is immediate.

Remark 1. If Sf and Kf have equivalent L^p norms for $1 \leq p \leq \infty$ it is clear from our result that so do $S^{\#}f$ and $K^{\#}f$.
Also, when we choose $Kf = f$ in Theorem 2 we obtain the transfer principle of A. P. Calderón[1] as a corollary.

Furthermore, $Sf = f$ in Theorem 2 implies the existence of a positive constant C such that

$$\|f\|_p \leq C\|K^\# f\|_p$$

for all $f \in L^p(X)$, $1 < p < \infty$. This means that our result can also be used to transfer L^p norm inequalities to ergodic theory from harmonic analysis. This allows one to use the methods of harmonic analysis to study the reverse L^p norm inequalities in ergodic theory.

Note that even though we have adapted the argument of A. P. Calderón[1] to prove our result it is clear that our result is much more general and useful than Calderón transfer principle.

An Application: It is clear from the argument of our proof that in the definition of the operator S or K the supremum can be replaced with the limit.
If we set

$$Sf(t) = \lim_{\varepsilon \to \infty} \int_{\varepsilon}^{\infty} \frac{f(t+u) - f(t-u)}{u} du$$

then $S^\#$ is the ergodic Hilbert transform.
Similarly, if we set

$$Kf(t) = \sup_s \left| \frac{1}{s} \int_0^s f(t+u)du \right|,$$

then $K^\#$ is the ergodic maximal function. Note also that S is the classical Hilbert transform and K is the Hardy-Littlewood maximal function. On the other hand, we know that H^1 norm of a function $f \in L^1(\mathbb{R})$ is equal to the L^1 norm of K, and it is well known that Hardy space H^1 can be characterized by the Hilbert transform and

$$\|f\|_{H^1} \sim \|f\|_1 + \|Sf\|_1 .$$

It is also known that there exists a constant $C > 0$ such that

$$\|f\|_{H^1} \leq C\|Kf\|_1$$

thus we have

$$\|Sf\|_1 \leq C\|Kf\|_1$$

for all $f \in L^1(\mathbb{R})$. When we now use Theorem 2 we see that

$$\|S^\# f\|_1 \leq C\|K^\# f\|_1$$

for all $f \in L^1(X)$.

REFERENCES

1. A. P. Calderón, Ergodic theory and translation-invariant operators, *Proc. Nat. Acad. Sci.* USA. 59, 349-353 (1968).