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# A Vector-valued Weak Type Inequality for Ergodic Square Function

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**ABSTRACT**

Let $(X,\beta,\mu,\tau)$ be an ergodic, measure preserving dynamical system with $(X,\beta,\mu,\tau)$ a totally $\sigma$-finite measure space. Define the ergodic square function

$$Sf=\left(\sum_{k=1}^{\infty}\left|\frac{1}{n_{k+1}}\sum_{k=0}^{n_{k+1}-1}f(\tau^k(x)-\frac{1}{n_k}\sum_{k=0}^{n_k-1}f(\tau^k(x)\right|^2\right)^{1/2}$$

for and increasing sequence $(n_k)$ of positive integers. Then for $2\le r<\infty$ there exists a constant $C>0$ such that

$$\mu\left\{x:\left(\sum_j\left(Sf_j(x)\right)^r\right)^{1/r}>\lambda\right\}\le\frac{C_r}{\lambda}\sum_j\|f_j\|_1,$$

for all $f_j\in L^1(X), j=1,2,\ldots$, and for every $\lambda>0$.

**Mathematics Subject Classifications:** 47A35, 28D05.

**Keywords:** Ergodic Square Function, Vector-valued Inequality.

## INTRODUCTION

Vector-valued inequalities has long been studied in harmonic analysis but these inequalities have not been studied in ergodic theory extensively. In this article we show that vector-valued inequalities can also be studied in ergodic theory by proving a weak type vector-valued inequality for ergodic square function. The method we used is an extension of a method used to prove a usual weak type inequality for the ergodic square function and our research also shows how one can extend the well known methods to study the vector-valued inequalities in ergodic theory. It is also clear from our proof that our method can also be used to study the vector-valued inequalities in harmonic analysis.

We will need the following version of Calderón-Zygmund decomposition on $Z$ when proving our result. This lemma is Lemma 2.7 in R. L. Jones *et al.*[3] and therefore we don't include its proof here.

**Lemma 1.** For a function $f$ in $l^1(Z)$, and $\lambda > 0$. We can write $f = g + b$ where $g \in l^2(Z)$, and

(1) $\|g\|_{l^1} \leq \|f\|_{l^1}$,

(2) $\|g\|_\infty \leq 2\lambda$,

(3) $b = \sum_i b_i(x)$ where each $b_i$ satisfies:

(a) $b_i$ is supported on an interval $B_i$,

(b) $\sum_i b_i(j) = 0$ for each $i$,

(c) $\frac{1}{|B_i|}\sum_{j\in B_j} |b_i(j)| \leq 4\lambda$ and $\lambda \leq \frac{1}{|B_i|}\sum_{j\in B_j} |f(j)|$,

(d) $B_i \cap B_j = \phi$ for $i \neq j$.

Note that the above imply

$$\sum_j |B_i| \leq \frac{1}{\lambda}\sum_j \|b_i\|_{l^1} \leq \frac{1}{\lambda}\|f\|_{l^1}.$$

Also, if $\lambda \geq \|f\|_\infty$, then we take $f = g$ and $b = 0$.

Let $(X, \beta, \mu, \tau)$ be an ergodic, measure preserving dynamical system with $(X, \beta, \mu, \tau)$ a totally $\sigma$-finite measure space. Define the ergodic square function

$$Sf = \left(\sum_{k=1}^{\infty}\left|\frac{1}{n_{k+1}}\sum_{k=0}^{n_{k+1}-1} f(\tau^k(x) - \frac{1}{n_k}\sum_{k=0}^{n_k-1} f(\tau^k(x)\right|^2\right)^{1/2}$$

for and increasing sequence $(n_k)$ of positive integers. Then we have the following result:

**Theorem 1.** Let $(n_k)$ be an increasing sequence of positive integers, then for $2 \le r < \infty$ there exists a constant $C > 0$ such that

$$\mu\left\{x : \left(\sum_j \left(Sf_j(x)\right)^r\right)^{1/r} > \lambda\right\} \le \frac{C_r}{\lambda}\sum_j \|f_j\|_1,$$

for all $f_j \in L^1(X), j = 1,2, \ldots$, and for every $\lambda > 0$.

**Proof.** As in the proof of Theorem 2.6 of R. L. Jones *et al.,*[3] we will use the Calderón-Zygmund decomposition on $Z$ with the same notations as in Lemma 1.

Let now $\tilde{B}_j^i$ denote an interval of length $5|B_j^i|$ and with the same center as $B_j^i$. Let $B_j = \cup_i \tilde{B}_j^i$, and $\tilde{B} = \cup_j \tilde{B}_j$. Let $l \in \tilde{B}$, $l \notin \tilde{B}$. We have

$$\begin{aligned} Sb_j(l)^2 &= \sum_{k=1}^{\infty}\left|A_{n_{k+1}} b_j(l) - A_{n_k} b_j(l)\right|^2 \\ &= \sum_{k=1}^{\infty}\left|A_{n_{k+1}}\left(\sum_i b_j^i(l)\right) - A_{n_k}\left(\sum_i b_j^i(l)\right)\right|^2 \\ &= \sum_{k=1}^{\infty}\left|\sum_i \left(A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l)\right)\right|^2 . \end{aligned}$$

Note that for any $i$ for which the average includes all the points in $B_j^i$, the average 0 by (3b). Thus for each fixed $k$, $A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l)$ is non-zero only if $l+1 \in B_j^i$, i.e., at least the average starts in $B_j^i$. The first possibility, starting in $B_j^i$, is excluded since $l \notin \tilde{B}_j$. Hence for each fixed $k$ and $l$, $A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l)$ is non-zero at most 2 values of $i$, an ending value for $A_{n_{k+1}} b_j^i(l)$ and an ending value for $A_{n_k} b_j^i(l)$. Thus we have

$$\begin{aligned} Sb_j(l)^2 &\le 2\sum_{k=1}^{\infty}\sum_i \left|A_{n_{k+1}} b_j^i(l) - A_{n_k} b_j^i(l)\right|^2 \\ &= 2\sum_i Sb_j^i(l). \end{aligned}$$

Then we have

$$\#\left\{l:\left(\sum_j \left(Sb_j(l)\right)^r\right)^{1/r} > \lambda\right\} \leq \#\left\{l:\sum_j \left(Sb_j(l)\right)^2 > \lambda^2\right\}$$
$$=\#\left\{l: l\notin \tilde{B}, \sum_j \left(Sb_j(l)\right)^2 > \lambda^2\right\} + \#\left\{l: l\in \tilde{B}, \sum_j \left(Sb_j(l)\right)^2 > \lambda^2\right\}$$
$$\leq \frac{1}{\lambda^2}\sum_{l\notin\tilde{B}}\sum_j \left(Sb_j(l)\right)^2 + \left|\tilde{B}\right|.$$

We also have

$$\frac{1}{\lambda^2}\sum_{l\notin\tilde{B}} Sb_j(l)^2 \leq \frac{1}{\lambda^2}\sum_{l\notin\tilde{B}} 2\sum_i Sb_j^i(l)^2$$
$$\leq 2\sum_i \frac{1}{\lambda^2}\sum_{l\notin\tilde{B}_j^i} Sb_j^i(l)^2.$$

Let us now fix $j$, then we claim that for each $i$ we have

$$\frac{1}{\lambda^2}\sum_{l\notin\tilde{B}_j^i} Sb_j^i(l)^2 \leq C\left|B_j^i\right|.$$

We know that translation by an integer is measure preserving, thus we can assume, without loss of generality, that $B_j^i = [0, N-1]$ where $\left|B_j^i\right| = N$. Note that since we only need to consider $l \in \tilde{B}_j^{i\,c}$, we do not need to consider $l \in (-2N, 3N)$, and since we are only looking at forward averages, we only need to consider $l \in (-\infty, 2N]$. To have a non-zero value of $A_{n_{k+1}} b_j^i(l)$ we must reach the support of $b_j^i$. Hence, we must have $n_{k+1} + l \geq 0$. Thus, $n_{k+1} \geq |l|$. But we might have $n_k + l \geq 0$ or $n_k + l < 0$ for that particular value of $k$. Let $n(l)$ be the smallest integer such that $n_{n(l)+1} \geq |l|$. Then $n_{n(l)} + l < 0$ and so we have arranged $b_j^i(l+r) = 0$ for all $r = 1, \ldots, n_{n(1)}$.

$$
\begin{aligned}
Sb_j^i(l)^2 &= \sum_{n_{k+1}\geq|l|}\left|A_{n_{k+1}}b_j^i(l)-A_{n_k}b_j^i(l)\right|^2 \leq \left(\sum_{n_{k+1}\geq|l|}\left|A_{n_{k+1}}b_j^i(l)-A_{n_k}b_j^i(l)\right|\right)^2 \\
&\leq \left(\sum_{n_{k+1\geq|l|}}\left\{\left|\left(\frac{1}{n_{k+1}}-\frac{1}{n_k}\right)\sum_{r=1}^{n_k}b_j^i(l+r)\right|+\frac{1}{n_{k+1}}\sum_{r=n_k+1}^{n_k+1}\left|b_j^i(l+r)\right|\right\}\right)^2 \\
&\leq \left(\sum_{n_{k+1}\geq|l|}\left(\frac{1}{n_k}-\frac{1}{n_{k+1}}\right)\sum_{r=1}^{n_k}\left|b_j^i(l+r)\right|+\sum_{n_{k+1}\geq|l|}\frac{1}{n_{k+1}}\sum_{r=n_k+1}^{n_{k+1}}\left|b_j^i(l+r)\right|\right)^2 \\
&\leq 2\left(\sum_{n_{k+1}\geq|l|}\left(\frac{1}{n_k}-\frac{1}{n_{k+1}}\right)\sum_{r=1}^{n_k}\left|b_j^i(l+r)\right|\right)^2+2\left(\sum_{n_{k+1}\geq|l|}\frac{1}{n_{k+1}}\sum_{r=n_k+1}^{n_{k+1}}\left|b_j^i(l+r)\right|\right)^2 \\
&\leq 2\left(\sum_{k=n(l)+1}\left(\frac{1}{n_k}-\frac{1}{n_{k+1}}\right)\sum_{r=1}^{N-1_k}\left|b_j^i(r)\right|\right)^2+2\left(\sum_{k=n(l)}\sum_{r=n_k+1}^{n_{k+1}}\left|b_j^i(l+r)\right|\right) \\
&\leq 2\left(\frac{1}{n_{n(l)+1}}N\frac{1}{N}\sum_{r=0}^{N-1}\left|b_j^i(r)\right|\right)^2+2\left(\sum_{k=n(l)}^{\infty}\frac{1}{n_{n(l)+1}}\sum_{r=n_k+1}^{n_{k+1}}\left|b_j^i(l+r)\right|\right)^2 \\
&\leq 32\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2+32\left(\frac{1}{n_{n(l)+1}}N\frac{1}{N}\sum_{r=0}^{N-1}\left|b_j^i(r)\right|\right)^2 \\
&\leq 32\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2+32\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2 \\
&\leq 64\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\sum_{l\notin\tilde{B}_j^i}Sb_j^i(l)^2 &\leq \sum_{l\leq-2N}64\left(\frac{1}{n_{n(l)+1}}N\lambda\right)^2 \\
&\leq \sum 64\left(\frac{1}{l}N\lambda\right)^2 \\
&\leq 64N^2\lambda^2\sum_{l\leq-2N}\frac{1}{l^2} \\
&\leq 64\lambda^2 N \\
&= 64\lambda^2\left|B_j^i\right|,
\end{aligned}
$$

and this proves our claim.

We now use our claim to get

$$\#\left\{l:\left(\sum_j\left(Sb_j(l)\right)^r\right)^{1/r}>\frac{\lambda}{2}\right\}\leq\#\left\{l:\sum_j\left(Sb_j(l)\right)^2>\left(\frac{\lambda}{2}\right)^2\right\}$$

$$\leq\frac{4}{\lambda^2}\sum_{l\notin\tilde{B}_j}\sum_j\left(Sb_j(l)\right)^2+\left|\tilde{B}\right|$$

$$\leq\frac{4}{\lambda^2}\sum_i\sum_{l\notin\tilde{B}_j^i}\sum_j\left(Sb_j^i(l)\right)^2+\left|\tilde{B}\right|$$

$$\leq C\sum_j\sum_i\left|B_j^i\right|+\left|\tilde{B}\right|$$

$$\leq\frac{C+1}{\lambda}\sum_j\left\|f_j\right\|_{l^1},$$

since we also have

$$\left|\tilde{B}\right|\leq\sum_j\sum_i\left|B_j^i\right|\leq\frac{1}{\lambda}\sum_j\left\|f_j\right\|_{l^1},$$

since we also have

$$\#\left\{l:\left(\sum_j\left(Sg_j(l)\right)^r\right)^{1/r}>\frac{\lambda}{2}\right\}\leq\frac{C}{\lambda^2}\sum_l\left(\sum_j\left(Sg_j(l)\right)^r\right)^{2/r}$$

$$\leq\frac{C}{\lambda^2}\sum_l\sum_j\left(Sg_j(l)\right)^2$$

$$\leq\frac{C}{\lambda^2}\sum_j\sum_l\left|g_j(l)\right|^2$$

$$\leq\frac{C}{\lambda}\sum_j\left\|g_j\right\|_{l^1}$$

$$\leq\frac{C}{\lambda}\sum_j\left\|f_j\right\|_{l^1}.$$

Thus we have

$$\#\left\{l:\left(\sum_j\left(Sf_j(l)\right)^r\right)^{1/r}>\lambda\right\}\leq\frac{C}{\lambda}\sum_j\left\|f_j\right\|_{l^1}.$$

When we now apply the Calderón transfer principle to the last inequality we get the inequality of our theorem with the same constant.

**Remark 1.** This research consists of one of the results in author's Ph.D thesis[2].

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