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Inverse Linear Fractional Programming: A New Approach

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ABSTRACT

This paper proposed an inverse model for linear fractional programming (LFP) problem, where the coefficients in the objective function are adjusted as little as possible so that the given feasible solution x^0 and objective value z^0 becomes optimal. Here the parametric approach is used for formulating the inverse LFP problem as a linear programming problem. The method has been illustrated by a numerical example also.

Keywords: Inverse optimization, Linear Fractional Programming, Parametric Linear Program.

1. INTRODUCTION

In an optimization problem, there are some parameters associated with the decision variable in the objective function and constraint's set. When solving the problem, generally it is assumed that all the parameters are known and our object is to find the optimal solution. Practically it is difficult to determine all model parameters with precision but it is plausible to call a solution x^0 which is not optimal under the present parameters but if we adjust some or all model parameters then the given solution x^0 could be an optimal solution.

Burton and Toint[1] were the first who investigate the inverse optimization for shortest path problem under l_2 norm, since then a lot of work has been done on inverse optimization but most of the work is based on combinatorial optimization problems. Zhang and Liu[2] have first been calculated some inverse linear programming problem and further investigated inverse linear programming problems in[3]. Ahuja and Orlin[4] provide various references in the area of inverse optimization and compile several applications in network flow problems with unit weight and develop combinatorial proofs of correctness. Huang and Liu[5] and Amin and Emrouznejad[6], have considered applications of inverse problem. Yibing, Tiesong and Zhongping[7] worked on inverse optimal value problem. Zhang and Zhang[8-10] worked on inverse quadratic programming problems, and Wang[11] has given the cutting plane algorithm for inverse integer programming problem. Hladik[12] have first been considered inverse problem for generalized linear fractional programming. They have shown that how much data of a generalized linear fractional program can vary such that the optimal values do not exceed some prescribed bounds. Jaing, Xiao, Zhang and Zhang[13] has given the perturbation approach for a type of inverse linear programming problem.

The linear fractional programming problem seeks to optimize the objective function of non-negative variables of quotient form with linear functions in numerator and denominator subject to a set of linear and homogeneous constraints. Bajanilov[14] compiled the literature of Linear Fractional Programming: Theory, Methods, Applications and Software in the form of book. Dinkelbach[15], Charnes-Cooper[16], Kantiswarup[17], Chadha[18,19], Jain and Mangal[20,21], Jain, Mangal and Parihar[22], Borza, Rambely, and Saraj[23] and many researchers gave different methods for solving linear fractional programming problem.

In the following section, a model for inverse linear fractional programming problem has been described. The model is based on parametric approach, which is the most popular approach for fractional programming problem (not necessarily linear) given by Dinkelbach[15]. The following theorem which is the theoretical foundation of Dinkelbach algorithm, also play the key role in our discussion.

Theorem 1.1 Vector x^* is an optimal solution of LFP (1) if and only if

$$F(\lambda^*) = \max_{x \in S}\{C(x) - \lambda^* D(x)\} = 0$$

$$\text{Where } \lambda^* = \frac{C(x^*)}{D(x^*)}$$

In our proposed method, we fix a feasible solution x^0 and objective value z^0, then formulate the LFP as parametric LP form and obtain its dual. For obtaining the inverse problem; we adjust the parameters associated with objective function of LFP, as little as possible (under l_1 measure) and apply the optimality condition. Then by using some standard transformation, the inverse problem reduces to a linear programming problem having large number of variables. The reduced problem can be solved by the optimization software's like TORA, EXCEL SOLVER etc.

2. INVERSE PROBLEM FOR LINEAR FRACTIONAL PROGRAMMING

The general linear fractional programming problem is given as follows:

$$\text{Maximize } z = \frac{C(x)}{D(x)} = \left(\sum_{j \in J} c_j x_j + c_0\right) / \left(\sum_{j \in J} d_j x_j + d_0\right)$$

$$\text{Subject to, } \quad \sum_{j \in J} a_{ij}\, x_j \le b_i \quad \text{for all } i \in I$$

$$x_j \ge 0 \quad \text{for all } j \in J \quad (1)$$

Where I denote the index set of constraints and J is the index set of decision variables and also $\sum_{j\in J} d_j x_j + d_0 > 0$ in the feasible region.

Now we consider the following parametric LPP

Max $\{C(x) - \lambda D(x)\}$

or

Max $\{(\sum_{j\in J} c_j x_j + c_0) - \lambda(\sum_{j\in J} d_j x_j + d_0)\}$

s.t. $\sum_{j\in J} a_{ij} x_j \le b_i$ for all $i \in I$

$x_j \ge 0$ for all $j \in J$ (2)

where λ is a real number.

The dual of this LP it the following linear program:

Min $\sum_{i\in I} b_i y_i + (c_0 - \lambda d_0)$

s.t. $\sum_{i\in I} a_{ij} y_i \ge (c_j - \lambda d_j)$

$y_i \ge 0$ for all $i \in I$ (3)

The optimality condition for LP state that, x and y solve the respective primal and dual problem if and only if

(i) $\sum_{j\in J} a_{ij} x_j \le b_i, x_j \ge 0$
(primal feasibility)

(ii) $\sum_{i\in I} a_{ij} y_i \ge (c_j - \lambda d_j), y_i \ge 0$
(dual feasibility)

(iii) $(\sum_{j\in J} c_j x_j + c_0) - \lambda(\sum_{j\in J} d_j x_j + d_0)$
$= \sum_{i\in I} b_i y_i + (c_0 - \lambda d_0)$ (strong duality)

If x^0 is the given feasible solution of the LFP with the objective value z^0, then our object is to adjust the parameters associated with the objective function as little as possible so that the given solution become optimal. Let c' and d' are the adjusted values of parameters c and d respectively, and also $C'(x)$ and $D'(x)$ are the modified values of $C(x)$ and $D(x)$, then by theorem 1.1, x^0 will be an optimal solution with the objective value z^0, if

$F(z^0) = \max_{x\in S}\{C'(x) - z^0 D'(x)\} = 0$

Where $z^0 = \frac{C'(x^0)}{D'(x^0)}$ (4)

If we replace c, d, λ and x by c', d', z^0 and x^0 respectively and use the result (4), then the optimality conditions can be restate as:

$\sum_{i\in I} a_{ij} y_i \ge (c_j' - z^0 d_j')$

$\sum_{i\in I} b_i y_i + (c_0' - z^0 d_0') = 0 = \left(\sum_{j\in J} c_j' x_j^0 + c_0'\right) - z^0\left(\sum_{j\in J} d_j' x_j^0 + d_0'\right)$

$y_i \ge 0$ for all $i \in I$ (5)

Note: we are interested in the objective value only so it is assumed that the primal problem is feasible.

Now the inverse problem is to minimize $\|(c', d') - (c, d)\|$ so that the given feasible solution x^0 become optimal with the objective value z^0. Here $\|\cdot\|$ is some selected norm, which may be l_1, l_2 or l_∞. If we consider the l_1 norm, then the inverse problem can be formulated as:

Min $\sum_{j\in J\cup\{0\}}[|c_j' - c_j| + |d_j' - d_j|]$

s.t. $\sum_{i\in I} a_{ij} y_i \ge (c_j' - z^0 d_j')$

$\sum_{i\in I} b_i y_i + (c_0' - z^0 d_0') = 0$

$\left(\sum_{j\in J} c_j' x_j^0 + c_0'\right) - z^0\left(\sum_{j\in J} d_j' x_j^0 + d_0'\right) = 0$

$y_i \ge 0$ for all $i \in I$ (6)

Let us assume $c_j' = c_j + e_j - f_j$; $e_j, f_j \ge 0$ and $d_j' = d_j + p_j - q_j$; $p_j, q_j \ge 0$ for all $j \in J \cup \{0\}$ then the inverse problem will be given as

Min $\sum_{j\in J\cup\{0\}}\{e_j + f_j + p_j + q_j\}$

s.t. $\sum_{i\in I} a_{ij} y_i - e_j + f_j + z^0(p_j - q_j)$
$\ge (c_0 - z^0 d_0)$ for all $j \in J$

$\sum_{i\in I} b_i y_i + e_0 - f_0 - z^0(p_0 - q_0)$
$+(c_0 - z^0 d_0) = 0$

$$\sum_{j\in J}\{ e_j - f_j - z^0(p_j - q_j)\}x_j^0 + e_0 - f_0 - z^0(p_0 - q_0) + \sum_{j\in J}(c_j - z^0 d_j)\, x_j^0 + c_0 - z^0 d_0 = 0$$

$y_i \geq 0$ for all $i \in I$ and $e_j, f_j, p_j, q_j \geq 0$ for all $j \in J \cup \{0\}$ (7)

Particularly, if only the objective value z^0 is given and we wish to find the adjusted values of the parameters in objective function, so that the desired objective value can be obtained. This problem is called the inverse optimal value problem, which can be obtained by inverse problem (7), if we remove the third equation from it.

Note: It is not necessary that the solution of inverse problem may always exist. The existence of the solution of inverse problem depends on the given solution x^0 and optimal value z^0, because the objective value has certain lower and upper bounds. We are restricting our self for those situations where the solution of inverse problem will exist.

3. NUMERICAL EXAMPLE

Let us consider a LFP problem

$$\text{Max}(z) = \frac{5x_1 - 3x_2 + 2}{4x_1 + x_2 - 2}$$

Subject to, $-x_1 - 2x_2 \leq -4$

$x_1 + 3x_2 \leq 6$

and $x_1, x_2 \geq 0$

The optimal solution of this LFP is $x_1 = 4, x_2 = 0$ with the objective value z = 11/7 and if $x_1^0 = 3, x_2^0 = 0.5$ is the given feasible solution with the objective value z = 1, then the inverse problem under l_1 is the following linear program:

Min $\sum_{j=0}^{j=2}(e_j + f_j + p_j + q_j)$

s.t. $\sum_{i=1}^{i=2} a_{ij}\, y_i - e_j + f_j + \lambda^0(p_j - q_j) \geq (c_0 - z^0 d_0)$; j = 1, 2

$\sum_{i=1}^{i=2} b_i\, y_i + e_0 - f_0 - \lambda^0(p_0 - q_0) + (c_0 - z^0 d_0) = 0$; j = 1, 2

$\sum_{j=1}^{j=2}\{e_j - f_j - z^0(p_j - q_j)\}\, x_j^0 + e_0 - f_0 - z^0(p_0 - q_0) + \sum_{j=1}^{j=2}(c_j - z^0 d_j)x_j^0 + c_0 - z^0 d_0 = 0$; j = 1, 2

$y_i \geq 0$ for all i = 1, 2 and $e_j, f_j, p_j, q_j \geq 0$ for all j = 0, 1, 2

Substituting $a_{11} = -1$, $a_{12} = -2$, $a_{21} = 1$, $a_{22} = 3$, $b_1 = -4$, $b_2 = 6$, $c_0 = 2$, $c_1 = 5$, $c_2 = -3$, $d_0 = -2$, $d_1 = 4$, $d_2 = 1$, $z^0 = 1$ and simplifying, we have

Min $(e_0 + f_0 + e_1 + f_1 + e_2 + f_2 + p_0 + q_0 + p_1 + q_1 + p_2 + q_2)$

$-y_1 + y_2 - e_1 + f_1 + p_1 - q_1 \geq 1$

$2y_1 - 3y_2 + e_2 - f_2 - p_2 + q_2 \leq 4$

$4y_1 - 6y_2 - e_0 + f_0 + p_0 - q_0 = 4$

$e_0 - f_0 + 3e_1 - 3f_1 + 0.5e_2 - 0.5f_2 - p_0 + q_0 - 3p_1 + 3q_1 - 0.5p_2 + 0.5q_2 = -5$

$$y_1, y_2, e_0, f_0, e_1, f_1, e_2, f_2, p_0, q_0, p_1, q_1, p_2, q_2 \geq 0$$

Optimal solution using TORA: $y_1 = 1$, $f_1 = 2$, $e_2 = 2$, using these values the new values of c_1 and c_2 are 3 and -1 respectively, therefore the modified objective function is given by

$$\text{Max}(z) = \frac{3x_1 - x_2 + 2}{4x_1 + x_2 - 2}$$

Now solving the problem with modified objective function, we obtain $x_1 = 3, x_2 = 0.5$ is an optimal solution with the objective value z = 1.

Further, we consider an inverse optimal value problem, where the desired objective value is given and our object is to make it an optimal objective value. Let z = 10/7 i.e. 1.428571 is the desired objective value, and then by removing the last constraint from (7) and substituting the values from the given LFP along with z = 1.428571, we get the inverse problem as:

$$\begin{aligned}
\text{Min} \quad & (e_0 + f_0 + e_1 + f_1 + e_2 + f_2 + p_0 + q_0 + p_1 + q_1 + p_2 + q_2) \\
& y_1 - y_2 + e_1 - f_1 - 1.428571p_1 + 1.428571q_1 \leq 0.714286 \\
& 2y_1 - 3y_2 + e_2 - f_2 - 1.428571p_2 + 1.428571q_2 \leq 4.428571 \\
& 4y_1 - 6y_2 - e_0 + f_0 + 1.428571p_0 - 1.428571q_0 = 4.857143 \\
& y_1, y_2, e_0, f_0, e_1, f_1, e_2, f_2, p_0, q_0, p_1, q_1, p_2, q_2 \geq 0
\end{aligned}$$

The optimal solution of inverse problem using TORA is $y_1 = 1.214286$ and $p_1 = 0.35$, so the adjusted value of d_1 is 4.35. Using the adjusted value of parameter the modified value of objective function is given by

$$\text{Max}(z) = \frac{5x_1 - 3x_2 + 2}{4.35x_1 + x_2 - 2}$$

Solving the LFP having modified objective function, we get $x_1 = 4$, $x_2 = 0$ with objective value z = 1.428571, which is the desired objective value.

4. CONCLUSION

An inverse version of linear fractional programming problem has been studied here. The new approach is useful in the situation where the enterprise wants to work with certain efficiency or want to fulfill the sudden market demand with certain efficiency and available resources. An illustration observation used to demonstrate the advantage of the new approach. This approach can further be extended to the nonlinear fractional programming.

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