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Extremal Solution for System of Fractional Hybrid Functional Integral Equations

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ABSTRACT

In this paper we prove existence the solution for system of fractional hybrid functional integral equations in Banach Space. A hybrid fixed point theorem in Banach space are used for proving the main result.

Keywords: Banach Space, fixed point theorem, System of Fractional hybrid functional integral equations.

1. INTRODUCTION

In this paper, our purpose is to prove the existence the extremal solution for system of fractional hybrid functional integral equation (FHFIE) by fixed point theorem.
We consider the following FHFIE

$$\left.\begin{aligned} x(t) &= f\Big(t, x\big(\alpha_1(t)\big), x\big(\alpha_2(t)\big)\Big) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g\Big(t, s, y\big(\gamma_1(s)\big), y\big(\gamma_2(s)\big)\Big)}{(t-s)^{1-\xi}}\,ds \\ y(t) &= f\Big(t, y\big(\alpha_1(t)\big), y\big(\alpha_2(t)\big)\Big) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g\Big(t, s, x\big(\gamma_1(s)\big), x\big(\gamma_2(s)\big)\Big)}{(t-s)^{1-\xi}}\,ds \end{aligned}\right\},$$

$\forall\ t \in \mathbb{R}_+$ (1.1)

Where $\xi \in (0,1)$ and the functions
$f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, g: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ \alpha_i, \gamma_i: \mathbb{R} \to \mathbb{R}, i = 1,2$ are continuous.

By a solution of system of FHFIE (1.1) we mean a function $(x, y) \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R} \times \mathbb{R})$is the space of absolutely continuous real valued functions on $\mathbb{R}_+$.

Our method of study is to convert the FHFIE (1.1) into an operator equation and then apply the fixed point theorem in the Banach space under some suitable conditions on the nonlinearities f and g.

2. PRELIMINARIES

In this section, we introduce notations, definitions and theorems which will be used in the sequel.

Definition 2.1[5]: An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset S of $\mathbb{X}$, $\mathbb{Q}(S)$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition 2.2[5]: Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q}: \mathbb{X} \to \mathbb{X}$, be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

i. Compact if $\mathbb{Q}(X)$ is relatively compact subset of $\mathbb{X}$.

ii. Totally compact if $\mathbb{Q}(S)$ is totally bounded subset of $\mathbb{X}$ for any bounded subset S of $\mathbb{X}$.

iii. Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$.

Definition 2.3[6]: Let $f \in \mathcal{L}^1[0, \mathbb{T}]$and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ξ of real function f is defined as $\mathfrak{D}^{\xi} f(t) = \frac{1}{\Gamma(1-\xi)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\xi}} ds \quad , \quad 0 < \xi < 1$

Such that $\mathfrak{D}^{-\xi} f(t) = I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds$ respectively.

Definition 2.4[6]: The Riemann-Liouville fractional integral of order $\xi \in (0,1)$ of the function $f \in \mathcal{L}^1[0, \mathbb{T}]$ is defined by the formula: $I^{\xi} f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds, \quad t \in [0, \mathbb{T}]$

where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by $\mathfrak{D}^{\xi} = \frac{d^{\xi}}{dt^{\xi}} = \frac{d}{dt} \circ I^{1-\xi}$.It may be shown that the fractional integral operator I^{ξ} transforms the space $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ into itself and has some other properties.

Theorem 2.1[5]: (Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in$\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

Theorem 2.2[5]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem 2.3[5] (Lebesgue's dominated convergence theorem): Suppose that $\{g_n\}$ is a sequence of measurable functions, that $g_n \to g$ pointwise a.e. as $n \to \infty$, and that $|g_n| \leq f, \forall\, n$, where f is integrable then g is integrable and

$$\int g\, d\mu = lim_{n\to\infty} \int g_n\, d\mu$$

Definition 2.5[3,4]: A mapping $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is called dominating function or in short $\mathcal{D}$-function if it is continuous and monotonic nondecreasing function satisfies $\varphi(0) = 0$.

Definition 2.6[3,4]: Let $\mathbb{X}$ be a Banach algebra with norm $\|\cdot\|$. A mapping $G: \mathbb{E} \to \mathbb{E}$ is called $\mathcal{D}$-lipschitz or nonlinear $\mathcal{D}$- lipschitz, if there exist continuous nondecreasing function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|Gx - Gy\| \leq \varphi\|x - y\|$ for all $x, y \in \mathbb{X}$ where $\varphi(0) = 0$, if $\varphi(r) = kr, k > 0$, then G is called lipschitz with lipschitz constant k.
If $k < 1$, G is called contraction with contraction constant k.
Finally G is called nonlinear $\mathcal{D}$-contraction if it is nonlinear $\mathcal{D}$- lipschitz with $\varphi(r) < r$ for $r > 0$.
Now recall the definition of coupled fixed point for a bivariate mapping.

Definition 2.7[8]: An element $(x, y) \in \mathbb{X} \times \mathbb{X}$ is called coupled fixed point of a mapping $T: \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ if $T(x, y) = x$ and $T(y, x) = y$
By a solution of system of FHFIE (1.1) we mean a function $x, y \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R} \times \mathbb{R})$ such that:

i. The function $t \to x - f(t, x_1, x_2)$ is absolutely continuous for each $x_1, x_2 \in \mathbb{R}$, and
ii. (x, y) satisfies the system of equations in (1.1)

Where $\mathcal{AC}(\mathbb{R}_+, \mathbb{R} \times \mathbb{R})$ is the space of absolutely continuous real valued functions on $\mathbb{R}_+$.
The following lemma (2.1) we introduce the certain Banach space which is used in our result.

Lemma 2.1[8]: Let $\widetilde{\mathbb{X}} = \mathbb{X} \times \mathbb{X}$. Define $\|(x, y)\| = \|x\| + \|y\|$
Then $\widetilde{\mathbb{X}}$ is Banach space with respect to the above norm.

Definition 2.8[2]: A closed and non-empty set $\mathbb{K}$ in a Banach Algebra $\mathbb{X}$ is called a cone if

i. $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$
ii. $\lambda\mathbb{K} \subseteq \mathbb{K}$ for $\lambda \in \mathbb{R}, \lambda \geq 0$
iii. $\{-\mathbb{K}\} \cap \mathbb{K} = 0$ where 0 is the zero element of $\mathbb{X}$. and is called positive cone if
iv. $\mathbb{K} \circ \mathbb{K} \subseteq \mathbb{K}$

And the notation $\circ$ is a multiplication composition in $\mathbb{X}$
We introduce an order relation $\leq$ in $\mathbb{X}$ as follows.
Let $x, y \in \mathbb{X}$ then $x \leq y$ if and only if $y - x \in \mathbb{K}$. A cone $\mathbb{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathbb{K}$. It is known that if the cone $\mathbb{K}$ is normal in $\mathbb{X}$ then every order-bounded set in $\mathbb{X}$ is norm-bounded set in $\mathbb{X}$. We equip the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of continuous real valued function on $\mathbb{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$$\mathbb{K} = \{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}): x(t) \geq 0 \ \forall t \in \mathbb{R}_+\} \tag{2.1}$$

We well known that the cone $\mathbb{K}$ is normal and positive in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. As a result of positivity of the cone $\mathbb{K}$ we have:

Lemma 2.2[3]: Let $p_1, p_2, q_1, q_2 \in \mathbb{K}$ be such that $p_1 \leq q_1$ and $p_2 \leq q_2$ then $p_1 p_2 \leq q_1 q_2$.

For any $p, q \in \mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}), p \leq q$ the order interval $[p, q]$ is a set in $\mathbb{X}$ given by,

$$[p, q] = \{x \in \mathbb{X}: p \leq x \leq q\} \tag{2.2}$$

we prove the existence of extremal solutions of the equations (1.1) under certain monotonicity conditions by using following fixed pint theorem of Dhage[4].

Theorem 2.4[4] : Let $[p, q]$ be an ordered interval in a Banach Space $\mathbb{X}$. Suppose that $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{X}$ are two operators such that

a. $\mathbb{A}$ is a nonlinear $\mathcal{D}$- contraction with $\mathcal{D}$-function α,
b. $\mathbb{B}$ is completely continuous,
c. $\mathbb{A}x + \mathbb{B}x \in [p, q]$ for each $x \in [p, q]$ and
d. $\mathbb{A}$and$\mathbb{B}$ are nondecreasing.

Further if the cone $\mathbb{K}$ is normal then the operator equation $\mathbb{A}x + \mathbb{B}x = x$ has the maximal and minimal solution in $[p, q]$.

3. EXISTENCE THEORY

Let $\mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$equipped with the supremum norm. Clearly it is a Banach space with respect to point wise operations and the supremum norm.

Define scalar multiplication and a sum on $\mathbb{X} \times \mathbb{X}$ as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and $\alpha(x, y) = (\alpha x, \alpha y)$for $\alpha \in \mathbb{R}$. Then $\mathbb{X} \times \mathbb{X}$ is a vector space on $\mathbb{R}$.

Definition 3.1: The function $g: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Caratheodory if

i. The function $(t, s) \to g(t, s, x, y)$ is measurable for each $x, y \in \mathbb{R}$
ii. The function $(x, y) \to g(t, s, x, y)$ is continuous for almost each $(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$
iii. Further the Caratheodory function $\beta(t, s, x, y)$ is called $\mathcal{L}^1$Caratheodory if
iv. For each real number $q > 0$ there exist a function $h_q \in \mathcal{L}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ such that
v. $|g(t, s, x, y)| \le h_q(t, s) a.e., (t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$for all $x, y \in \mathbb{R}$ with $|x| \le q$ and $|y| \le q$
vi. Finally a Caratheodory function $g(t, s, x, y)$ is called $\mathcal{L}^1_{\mathbb{X}}$- Caratheodory if
vii. $|g(t, s, x, y)| \le h(t, s) a.e., (t, s) \in \mathbb{R}_+ \times \mathbb{R}_+$ for all $x, y \in \mathbb{R}$
viii. For convince the function h referred to as a bound function.

4. EXISTENCE OF EXTREMAL SOLUTIONS

We need following definitions in sequel.

Definition 4.1: A function $p \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called a **lower solution** of the FNDE (1.1) on $\mathbb{R}_+$ if the function $t \to p(t) - f\left(t, p\left(\alpha_1(t)\right), p\left(\alpha_2(t)\right)\right)$ is continuous and

$$p(t) \le f\left(t, p\left(\alpha_1(t)\right), p\left(\alpha_2(t)\right)\right) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{g\left(t, s, p\left(\gamma_1(s)\right), p\left(\gamma_2(s)\right)\right)}{(t-s)^{1-\xi}} ds \qquad (4.1)$$

Again a function $q \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called an **upper solution** of the FHFIE (1.1) on $\mathbb{R}_+$ if function the $t \to q(t) - f\left(t, q(\alpha_1(t)), q(\alpha_2(t))\right)$ is continuous and

$$q(t) \geq f\left(t, q(\alpha_1(t)), q(\alpha_2(t))\right) + \frac{1}{\Gamma(\xi)} \int_0^t \frac{g\left(t,s,q(\gamma_1(s)),q(\gamma_2(s))\right)}{(t-s)^{1-\xi}} ds \tag{4.2}$$

Definition 4.2: A solution x_M of the FHFIE (1.1) is said to be **maximal** if for any other solution x to FHFIE (1.1) one has $x(t) \leq x_M(t)$ for all t ∈ $\mathbb{R}_+$. Again a solution x_M of the FHFIE (1.1) is said to be **minimal** if $x_M(t) \leq x(t)$ for all t ∈ $\mathbb{R}_+$ where x is any solution of the FHFIE (1.1) on $\mathbb{R}_+$.

Definition 4.3: (Caratheodory case) A function $\sigma: \mathbb{R} \to \mathbb{R}$ is nondecreasing if $\sigma(x) \leq \sigma(y)\ \forall x, y \in \mathbb{R}$ for which $x \leq y$.Similarly, $\sigma(x)$ is increasing in x if $\sigma(x) < \sigma(y)\ \forall x, y \in \mathbb{R}$ for which $x < y$.

We consider the following assumptions:

$\mathfrak{B}_1$)The functions $f\left(t, x(\alpha_1(t)), x(\alpha_2(t))\right)$ and $g\left(t, s, x(\gamma_1(s)), x(\gamma_2(s))\right)$ are nondecreasing in $x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ almost everywhere for $t \in \mathbb{R}_+$.

$\mathfrak{B}_2$) The operators $\mathbb{A}$ and $\mathbb{B}$ are nondecreasing on $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$.

$\mathfrak{B}_3$)The FHFIE (1.1) has a lower solution p and an upper solution q on $\mathbb{R}_+$with $p \leq q$.

$\mathfrak{B}_4$)The function $l: \mathbb{R}_+ \to \mathbb{R}$ defined by

$$l(t) = \left|g\left(t, s, p(\gamma_1(s)), p(\gamma_2(s))\right)\right| + \left|g\left(t, s, q(\gamma_1(s)), q(\gamma_2(s))\right)\right|$$

is Lebesgue measurable

Remark 4.1: Assume that ($\mathfrak{B}_2 - \mathfrak{B}_4$) hold. Then

$\left|g\left(t, x(t), x(\mu(t))\right)\right| \leq l(t), a.e.\ t \in \mathbb{R}_+$, for all $x \in [p, q]$.

Theorem 4.1: Suppose that the assumptions ($\mathfrak{B}_1 - \mathfrak{B}_4$) holds and l is given in remark (4.1) then FHFIE (1.1) has a minimal and maximal solution on $\mathbb{R}_+$.

Proof: Let $\mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and we define an order relation "≤" by the cone $\mathbb{K}$ given by (2.1). Clearly $\mathbb{K}$ is a normal cone in $\mathbb{X}$

Define two operators $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{K}$ by,

$$\mathbb{A}x(t) = f\left(t, x(\alpha_1(t)), x(\alpha_2(t))\right) \tag{4.1}$$

$$\mathbb{B}x(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{g\left(t,s,x(\gamma_1(s)),x(\gamma_2(s))\right)}{(t-s)^{1-\xi}} ds \tag{4.2}$$

So the system (1.1) transformed into the system of operator equation as

$$\left.\begin{aligned} x(t) &= \mathbb{A}x(t) + \mathbb{B}x(t) \\ y(t) &= \mathbb{A}y(t) + \mathbb{B}y(t) \end{aligned}\right\} t \in \mathbb{R}_+, \tag{4.3}$$

We shall show that the operators $\mathbb{A}$ and $\mathbb{B}$ satisfy all the conditions of theorem (2.4)

Notice that $\mathbb{A}, \mathbb{B}: [\mathcal{p}, \mathcal{q}] \to \mathbb{K}$ Since the cone $\mathbb{K}$ in $\mathbb{X}$ is normal, $[\mathcal{p}, \mathcal{q}]$is a norm bounded set in $\mathbb{X}$. Now it is shown, as in the proof of Theorem (4.1) in the paper[8], that $\mathbb{A}$ is a contraction with a contraction constant $\|\alpha\|$ and $\mathbb{B}$ is completely continuous operator on $[\mathcal{p}, \mathcal{q}]$. Again the hypothesis $(\mathfrak{B}2)$ implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing on $[\mathcal{p}, \mathcal{q}]$.

Step I: To show that the operators $\mathbb{A}, \mathbb{B}$and $\mathbb{C}$ are non-decreasing on $[\mathcal{p}, \mathcal{q}]$. let $x, y \in [\mathcal{p}, \mathcal{q}]$ be such that $x \le y$. $\therefore \mathbb{A}x(t) = f\left(t, x(\alpha_1(t)), x(\alpha_2(t))\right)$

$$\le f\left(t, y(\alpha_1(t)), y(\alpha_2(t))\right)$$
$$\le \mathbb{A}y(t), \forall t \in \mathbb{R}_+$$

Similarly,

$$\mathbb{B}x(t) = \frac{1}{\Gamma(\xi)}\int_0^t \frac{g\left(t,s,x(\gamma_1(s)),x(\gamma_2(s))\right)}{(t-s)^{1-\xi}}ds$$
$$\le \frac{1}{\Gamma(\xi)}\int_0^t \frac{g\left(t,s,y(\gamma_1(s)),y(\gamma_2(s))\right)}{(t-s)^{1-\xi}}ds$$
$$\le \mathbb{B}y(t), \forall t \in \mathbb{R}_+$$

Implies that $\mathbb{A}$ and $\mathbb{B}$ are nondecreasing operators on $[\mathcal{p}, \mathcal{q}]$.

Step II: Again definition (4.1) and hypothesis $(\mathfrak{B}3)$ implies that

$$\mathcal{p}(t) \le f\left(t, \mathcal{p}(\alpha_1(t)), \mathcal{p}(\alpha_2(t))\right) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g\left(t,s,\mathcal{p}(\gamma_1(s)),\mathcal{p}(\gamma_2(s))\right)}{(t-s)^{1-\xi}}ds$$
$$\le f\left(t, x(\alpha_1(t)), x(\alpha_2(t))\right) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g\left(t,s,x(\gamma_1(s)),x(\gamma_2(s))\right)}{(t-s)^{1-\xi}}ds$$
$$\le f\left(t, \mathcal{q}(\alpha_1(t)), \mathcal{q}(\alpha_2(t))\right) + \frac{1}{\Gamma(\xi)}\int_0^t \frac{g\left(t,s,\mathcal{q}(\gamma_1(s)),\mathcal{q}(\gamma_2(s))\right)}{(t-s)^{1-\xi}}ds$$
$$\le \mathcal{q}(t), \forall t \in \mathbb{R}_+ \text{and} x \in [\mathcal{p}, \mathcal{q}]$$

As a result $\mathcal{p}(t) \le \mathbb{A}x(t)\mathbb{B}x(t) \le \mathcal{q}(t), \forall t \in \mathbb{R}_+$ and $x \in [\mathcal{p}, \mathcal{q}]$
Hence $\mathbb{A}x\mathbb{B}x \in [\mathcal{p}, \mathcal{q}], \forall x \in [\mathcal{p}, \mathcal{q}]$

Now we apply theorem (2.4) to the operator equation $\mathbb{A}x\mathbb{B}x = x$ to yield that the FHFIE (1.1) has minimum and maximum solution on $\mathbb{R}_+$.
This completes the proof.

CONCLUSION

In this paper we have studied the existence the extremal solution for the system of fractional hybrid functional integral equation. The result has been obtained by using fixed point theorem for two operators in Banach space.

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