JOURNAL OF COMPUTER AND MATHEMATICAL SCIENCES
An International Open Free Access, Peer Reviewed Research Journal
www.compmath-journal.org

ISSN 0976-5727 (Print)
ISSN 2319-8133 (Online)
Abbr:J.Comp.&Math.Sci.
2014, Vol.5(4): Pg.382-389

Integration Involving Certain Products and I-Function Function of Two Variables

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(Received on: August 13, 2014)

ABSTRACT

In this paper, we evaluate some integrals involving the products of I-function of two variables and other hypergeometric functions using E-operator. In section (3.3), we evaluate some integrals involving the products of I-function of two variables and other hypergeometric functions, while in section (3.4), some integrals involving the product of generalized hypergeometric function and I-function of of two variables have been derived by means of finite difference operator E.

Keywords: Finite difference operator E, generalized hypergeometric function, double hypergeometric function.

3.1 INTRODUCTION

3.1.1. Definition of I Function:

The I-function of one variable introduced by Saxena[2], will be represented as follows:

$$\begin{matrix} m, n & [(a_j, \alpha_j)_{1, n}], [(a_{ji}, \alpha_{ji})_{n+1}, p_i] \\ p_i, q_i: r & [(b_j, \beta_j)_{1, m}], [(b_{ji}, \beta_{ji})_{m+1}, q_i] \end{matrix}$$

$$I^{m, n}_{p_i, q_i: r}\left[x \middle| \begin{matrix} [(a_j, \alpha_j)_{1, n}], [(a_{ji}, \alpha_{ji})_{n+1}, p_i] \\ [(b_j, \beta_j)_{1, m}], [(b_{ji}, \beta_{ji})_{m+1}, q_i] \end{matrix}\right] = (1/2\pi i) \int_L \theta(s)\, x^s\, ds \qquad (1.3.1)$$

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\}},$$

integral is convergent, when (B >0, A ≥ 0), where

$$B = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji}, \tag{1.3.2}$$

$$A = \sum_{j=1}^{p_i} \alpha_{ji} - \sum_{j=1}^{q_i} \beta_{ji} ,$$

$|\arg x| < \frac{1}{2} B\pi$, $\forall\, i \in (1, 2, \ldots, r)$. p_i (i = 1, 2, …, r); q_i(i = 1, 2, …, r); m, n are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i$, (i = 1, 2, …, r); r is finite α_j, β_j, α_{ji}, β_{ji} are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers such that

$\alpha_j (b_h + v) \ne B_h (a_j - 1 - k)$,

for v, k = 0, 1, 2, ….. h = 1, 2, …, m; j = 1, 2, …, r; L is a contour runs from $\sigma - i\infty$ to $\sigma + i\infty$ (σ is real), in the complex s-plane such that the poles of

$s = (a_j - 1 - v) \mid \alpha_j$

j = 1, 2, …, n; v = 0, 1, 2, ….

lie to the left hand side

$s = (b_j + v) \mid \beta_j$

j = 1, 2, …, m; v = 0, 1, 2, ….

and right of L.

The I–function of two variables introduced by Sharma & Mishra[9], will be defined and represented as follows:

$$I[\;] = I^{0,\, n\; :\, m_1,\, n_1\; :\, m_2,\, n_2}_{p_i,\, q_i :\, r\, :\, p_{i'},\, q_{i'} :\, r'\, : p_{i''},\, q_{i''} :\, r''} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} [(a_j;\alpha_j, A_j)_{1,n}], [(a_{ji};\alpha_{ji}, A_{ji})_{n+1, p_i}] \\ [(b_{ji};\beta_{ji}, B_{ji})_{1, q_i}] \end{matrix} \right.$$

$$\begin{matrix} : [(c_j;\gamma_j)_{1,n_1}], [(c_{ji'};\gamma_{ji'})_{n_1+1, p_{i'}}]; [(e_j;E_j)_{1,n_2}], [(e_{ji''};E_{ji''})_{n_2+1, p_{i''}}] \\ : [(d_j;\delta_j)_{1,m_1}], [(d_{ji'};\delta_{ji'})_{m_1+1, q_{i'}}]; [(f_j;F_j)_{1,m_2}], [(f_{ji''};F_{ji''})_{m_2+1, q_{i''}}] \end{matrix}$$

$$= \int_{L_1}\int_{L_2} \phi_1(\xi, \eta)\, \theta_2(\xi)\, \theta_3(\eta) x^{\xi} y^{\eta}\, d\xi\, d\eta, \tag{1.3.3}$$

where

$$\phi_1(\xi, \eta) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j\xi + A_j\eta)}{\sum_{i=1}^{r} \left[\prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji}\xi - A_{ji}\eta) \prod_{j=1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji}\xi + B_{ji}\eta) \right.} ,$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1} \Gamma(d_j - \delta_j\xi) \prod_{j=1}^{n_1} \Gamma(1 - c_j + \gamma_j\xi)}{\sum_{i'=1}^{r'} \left[\prod_{j=m_1+1}^{q_{i'}} \Gamma(1 - d_{ji'} + \delta_{ji'}\xi) \prod_{j=n_1+1}^{p_{i'}} \Gamma(c_{ji'} - \gamma_{ji'}\xi) \right]} ,$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_2} \Gamma(f_j - F_j\eta) \prod_{j=1}^{n_2} \Gamma(1 - e_j + E_j\eta)}{\sum_{i''=1}^{r''} \left[\prod_{j=m_2+1}^{q_{i''}} \Gamma(1 - f_{ji''} + F_{ji''}\eta) \prod_{j=n_2+1}^{p_{i''}} \Gamma(e_{ji''} - E_{ji''}\eta) \right]},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, $p_{i'}$, $p_{i''}$, q_i, $q_{i'}$, $q_{i''}$, n, n_1, n_2, n_j and m_k are non negative integers such that $p_i \geq n \geq 0$, $p_{i'} \geq n_1 \geq 0$, $p_{i''} \geq n_2 \geq 0$, $q_i > 0$, $q_{i'} > 0$, $q_{i''} > 0$, (i = 1, …, r; i′ = 1, …, r′; i′′ = 1, …, r′′; k = 1, 2) also all the A's, α's, B's, β's, γ's, δ's, E's and F's are assumed to be positive quantities for standardization purpose; the definition of I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the ξ–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \delta_j\xi)$ (j = 1, ..., m_1) lie to the right, and the poles of $\Gamma(1 - c_j + \gamma_j\xi)$ (j = 1, ..., n_1), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n) to the left of the contour.

The contour L_2 is in the η–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ (j=1,....., n_2) lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ (j = 1, ..., m_2), $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ (j = 1, ..., n) to the left of the contour. Also

$$R = \Sigma\, \alpha_{ji} + \Sigma\, \gamma_{ji'} - \Sigma\, \beta_{ji} - \Sigma\, \delta_{ji'} < 0,$$

$$S = \sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{p_{i''}} E_{ji''} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{q_{i''}} F_{ji''} < 0,$$

$$U = \sum_{j=n+1}^{p_i} \alpha_{ji} - \sum_{j=1}^{q_i} \beta_{ji} + \sum_{j=1}^{m_1} \delta_j - \sum_{j=m_1+1}^{q_{i'}} \delta_{ji'} + \sum_{j=1}^{n_1} \gamma_j - \sum_{j=n_1+1}^{p_{i'}} \gamma_{ji'} > 0, \quad (1.3.4)$$

$$V = -\sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{m_2} F_j - \sum_{j=m_2+1}^{q_{i''}} F_{ji''} + \sum_{j=1}^{n_2} E_j - \sum_{j=n_2+1}^{p_{i''}} E_{ji''} > 0, \quad (1.3.5)$$

and $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$.

3.1.2 REVIEW OF LITRATURE

Srivastava Renu and Srivastava S. S.[1,2], Tiwari, I. P. and Sharma C. K.[3,4], Shrivastava, H. M.[5], Burchnall J. L. and Chaundy T. Y.[6], Appell, P. and Kampe de Feriet J[7], Bromwich T. J. I. A.[8], Erdelyi A.[9] and several other authors have evaluated some integrals involving the product of I-function and other commonly used hypergeometric functions.

3.2 FORMULA USED

From Shrivastava[5] (with z replaced by iz are required in the present work:

$$z^{\lambda}F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');}-x^2z^2,-y^2z^2\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{n!}J_{\lambda+2n}(2z)\,F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');}x^2,y^2\right] \tag{3.2.1}$$

and

$$z^{\lambda}F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');}-x^2z^2,-y^2z^2\right]$$

$$=\Gamma(1+\lambda)\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}J_{\lambda+n}(2z)\,F\left[{}^{-n,1+\lambda,(a),(a');(c);(c');}_{(b),(b');(d);(d');}x^2,y^2\right], \tag{3.2.2}$$

where A + A′ + C ≤ B + B′ + D, A + A′ + C′ ≤ B + B′ + D′, and for all values of λ with possible exception of zero and negative integers. (a) represents the sequence of A parameters $a_1, a_2, \ldots, a_A$ and this convention will be retained throughout this chapter. The notation for double hypergeometric function is due to Burchnall and Chaundy[6] in preference, for the sake of brevity, to an earlier one introduced by Kampe de Feriet[7].

The finite difference operator E[8] has the following operations

$$E_a f(a) = f(a+1),\ E_a^n f(a) = E_a[E_a^{n-1} f(a)]. \tag{3.2.3}$$

3.3 MAIN INTEGRALS:

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$$\int_0^{\infty} z^{\rho+\lambda-1}\sin 2z\,F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');}-x^2z^2,-y^2z^2\right]I\left[{}^{\xi z^{-2m}}_{\eta}\right]dz$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n}n!}F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');}x^2,y^2\right]$$

$$I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+3:r';p_i'',q_i'':r''}\left[{}^{z^{2m}\xi}_{\eta}\Big|\begin{matrix}\ldots\ldots,\ldots\ldots:(1/2+\rho,2m),\ldots\ldots:\ldots\ldots,\ldots\ldots\\ \ldots\ldots,\ldots\ldots:\left(\frac{1}{2}+n+\frac{\lambda}{2}+\frac{\rho}{2},m\right),\ldots\ldots,\\ (-2n-\lambda+\rho,2m),(n+\lambda/2+\rho/2,m):\ldots\ldots,\ldots\ldots\end{matrix}\right], \tag{3.3.1}$$

which is valid under the conditions A + A′ + C ≤ B + B′ + D, A + A′ + C′ ≤ B + B′ + D′, $R(\rho+\lambda+\frac{2md_j}{\delta_j}) > -1$ (j = 1, …., k), $R(\rho+\lambda+\frac{2m(c_j-1)}{\gamma_j}) < 1$ (j = 1, …., *l*) and $U>0, V>0, |\arg\xi|<\frac{1}{2}U\pi$, where U and V are given in (1.3.4) and (1.3.5).

Proof of (3.3.1):

To prove (3.3.1), take the expansion (3.2.1), multiply both side by f(z), integrate with respect to z between the limits 0 to ∞ and interchange the order of integration and summation we get

$$\int_0^{\infty} z^{\lambda}F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');}-x^2z^2,-y^2z^2\right]f(z)dz$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda+2n)\Gamma(\lambda+n)}{n!}F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');}x^2,y^2\right]$$

$$.\int_0^{\infty} J_{\lambda+2n}(2z)\,f(z)dz, \tag{3.3.3}$$

for A + A′ + C ≤ B + B′ + D, A + A′ + C′ ≤ B + B′ + D′, R(λ + η + 1) > 0 and R(λ + ξ + 1) > 0, where $f(z) = O(|z|^{\eta})$, for small z and $f(z) = O(|z|^{\xi})$, for large z.

The change of integration and summation is justified [8] because

(i) the series

$$\sum_{N=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{n!} J_{\lambda+2n}(2z) \; F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right]$$

is uniformly convergent in $0 \leq z \leq N$, N being arbitrary;

(ii) f(z) is a continuous function of z for all value of $z \geq z_0 > 0$;

(iii) the integral on the left of (3.3.3) converges absolutely under the stated conditions.

Now on taking

$$f(z) = z^{\rho-1} \sin 2z \; I\left[{}^{\xi z^{-2m}}_{\eta}\right]$$

in (3.3.3), replacing I-function of two variables on the right by its equivalent contour integral as given in (1.3.3), changing the order of integration which is justified due to the absolute convergence of the integrals, evaluating the inner integral with the help of [9] and interpreting it with (1.3.3), we get (3.3.1).

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$$\int_0^{\infty} z^{\rho+\lambda-1} \cos 2z \; F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2z^2, -y^2z^2\right] I\left[{}^{\xi z^{-2m}}_{\eta}\right] dz$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda+2n)\Gamma(\lambda+n)}{2^{1+\rho-\lambda-2n} n!} F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right] I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+3:r';p_i'',q_i'':r''}$$

$$\left[{}^{z^{2m}\xi}_{\eta}\Big|{}^{\ldots\ldots,\ldots\ldots:(1/2+\rho,2m),\ldots\ldots:\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:\left(n+\frac{\lambda}{2}+\frac{\rho}{2},m\right),\ldots\ldots,(-2n-\lambda+\rho,2m),(n+\lambda/2+\rho/2,m):\ldots\ldots,\ldots\ldots}\right], \qquad (3.3.2)$$

which is valid under the conditions A + A′ + C ≤ B + B′ + D, A + A′ + C′ ≤ B + B′ + D′, $R\left(\rho+\lambda+\frac{2md_j}{\delta_j}\right) > 0$ (j = 1, …., k), $R\left(\rho+\lambda+\frac{2m(c_j-1)}{\gamma_j}\right) < 1$ (j = 1, …., *l*) and $U > 0, V > 0, |\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.3.4) and (1.3.5).

Proof of 3.3.2:

If we take

$$f(z) = z^{\rho-1} \cos 2z \; I\left[{}^{\xi z^{-2m}}_{\eta}\right]$$

proceed on the parallel lines as mentioned above and then in the light of the result[9], we obtain (3.3.2).

On considering the result (3.2.2), proceeding on the parallel lines as mentioned above and making use of the result [9, p.328(10); p.328(11)], we get the following different forms of the integral (3.3.1) and (3.3.2) as

$$\int_0^{\infty} z^{\rho+\lambda-1} \sin 2z \; F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2z^2, -y^2z^2\right] I\left[{}^{\xi z^{-2m}}_{\eta}\right] dz$$

$$= \frac{\Gamma(\lambda+1)}{2^{1+\rho-\lambda}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right] I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+3:r';p_i'',q_i'':r''}$$

$$\left[{}^{z^{2m}\xi}_{\eta}\Big|{}^{\ldots\ldots,\ldots\ldots:\left(\frac{1}{2}+\rho+n,2m\right),\ldots\ldots:\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:\left(\frac{1}{2}+n+\frac{\lambda}{2}+\frac{\rho}{2},m\right),\ldots\ldots,(-\lambda+\rho,2m),(n+\lambda/2+\rho/2,m):\ldots\ldots,\ldots\ldots}\right], \qquad (3.3.4)$$

which is valid under the same conditions as (3.3.1) and

$$\int_0^{\infty} z^{\rho+\lambda-1} \cos 2z \; F\left[{}^{(a),(a');(c);(c');}_{(b),(b');(d);(d');} - x^2z^2, -y^2z^2\right] I\left[{}^{\xi z^{-2m}}_{\eta}\right] dz$$

$$= \frac{\Gamma(\lambda+1)}{2^{1+\rho-\lambda}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} F\left[{}^{-n,\lambda+n,(a),(a');(c);(c');}_{(b),(b');(d);(d');} x^2, y^2\right]$$

$$I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+3:r';p_i'',q_i'':r''}$$

$$[{}^{z^{2m}\xi}_{\eta}|^{\ldots\ldots,\ldots\ldots:(\frac{1}{2}+\rho+n,2m),\ldots\ldots:\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:\left(n+\frac{\lambda}{2}+\frac{\rho}{2},m\right),\ldots\ldots,\ (-\lambda+\rho,2m),(n+\lambda/2+\rho/2,m):\ldots\ldots,\ldots\ldots}], \quad (3.3.5)$$

The conditions of validity for (3.3.5) are the same as for (3.3.2).

3.4 INTEGRALS BY MEANS OF FINITE DIFFERENCE OPERATOR E:

In this section we evaluate some integrals by means of finite difference operator E:

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, I[{}^{\xi\sin^{2h}\theta\cos^{2k}\theta}_{\eta}]$$

$$\cdot\ {}_uF_v\,[e_u; f_v; c \sin^{2\mu}\theta \cos^{2\nu}\theta]\, d\theta$$

$$= \left(\frac{1}{2}\right)\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!} I^{0,n_1;m_2,n_2+2;m_3,n_3}_{p_i,q_i:r;p_i'+2,q_i'+1:r';p_i'',q_i'':r''}$$

$$[{}^{\xi}_{\eta}|^{\ldots\ldots,\ldots\ldots:(1/2-\rho-\mu r,h),(1/2-\sigma-\nu r,k),\ldots\ldots:\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:\ldots\ldots,(-\rho-\sigma-(\mu+\nu)r,h+k):\ldots\ldots,\ldots\ldots}], \quad (3.4.1)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in H-function of two variables which is convergent if

$U = -\ \Sigma\ \alpha_j - \Sigma\ \beta_j - \Sigma\ \delta_j - \Sigma\ \delta_j + \Sigma\,\gamma_j - \Sigma\ \gamma_j > 0,$ (1.2.40)

$V = -\ \Sigma\ A_j - \Sigma\ B_j - \Sigma\ F_j - \Sigma\ F_j + \Sigma\ E_j - \Sigma\ E_j > 0,$ (1.2.41)

and $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$.

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, I[{}^{\xi\sin^{-2h}\theta\cos^{2k}\theta}_{\eta}]$$

$$\cdot\ {}_uF_v\,[e_u; f_v; c \sin^{2\mu}\theta \cos^{2\nu}\theta]\, d\theta$$

$$= \left(\frac{1}{2}\right)\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!} I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+2,q_i'+1:r';p_i'',q_i'':r''}$$

$$[{}^{\xi}_{\eta}|^{\ldots\ldots,\ldots\ldots:(1/2-\sigma-\nu r,k),\ldots\ldots,(1+\rho+\sigma+(\mu+\nu)r,h-k):\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:(1/2+\rho+\mu r,h),\ldots\ldots:\ldots\ldots,\ldots\ldots}], \quad (3.4.2)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.2.40) and (1.2.41).

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, I[{}^{\xi\sin^{2h}\theta\cos^{-2k}\theta}_{\eta}]$$

$$\cdot\ {}_uF_v\,[e_u; f_v; c \sin^{2\mu}\theta \cos^{2\nu}\theta]\, d\theta$$

$$= \left(\frac{1}{2}\right)\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!} I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+2:r';p_i'',q_i'':r''}$$

$$[{}^{\xi}_{\eta}|^{\ldots\ldots,\ldots\ldots:(1/2-\rho-\mu r,h),\ldots\ldots:\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:(1/2+\sigma+\nu r,k),\ldots\ldots,(-\rho-\sigma-(\mu+\nu)r,h-k):\ldots\ldots,\ldots\ldots}], \quad (3.4.3)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.2.40) and (1.2.41).

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, I[{}^{\xi\sin^{-2h}\theta\cos^{-2k}\theta}_{\eta}]$$

$$\cdot\ {}_uF_v\,[e_u; f_v; c \sin^{2\mu}\theta \cos^{2\nu}\theta]\, d\theta$$

$$= \left(\frac{1}{2}\right)\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{u}(e_j,r)c^r}{\prod_{j=1}^{v}(f_j,r)r!} I^{0,n_1;m_2+2,n_2;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+2:r';p_i'',q_i'':r''}$$

$$[{}^{\xi}_{\eta}|^{\ldots\ldots,\ldots\ldots:\ldots\ldots,(1+\rho+\sigma+(\mu+\nu)r,h+k):\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:(1/2+\rho+\mu r,h),(1/2+\sigma+\nu r,k),\ldots\ldots:\ldots\ldots,\ldots\ldots}], \quad (3.4.4)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.2.40) and (1.2.41).

Proof of (3.4.1):

As we know by the definite integral,

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta\, I[{}^{\xi\sin^{2h}\theta\cos^{2k}\theta}_{\eta}]\, d\theta$$

$$= (1/2)\, I^{0,n_1;m_2,n_2+2;m_3,n_3}_{p_i,q_i:r;p_i'+2,q_i'+1:r';p_i'',q_i'':r''}$$

$$[{}^{\xi}_{\eta}|^{\ldots\ldots,\ldots\ldots:(1/2-\rho,h),(1/2-\sigma,k),\ldots\ldots:\ldots\ldots,\ldots\ldots}_{\ldots\ldots,\ldots\ldots:\ldots\ldots,(-\rho-\sigma,h+k):\ldots\ldots,\ldots\ldots}], \quad (2.3.18)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.3.4) and (1.3.5).

On multiplying both side of (2.3.18) by $\frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda)c^{\lambda}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda)}$ and then applying the operator

$e^{E_\rho^\mu E_\sigma^\nu E_\lambda}$, we get

$$e^{E_\rho^\mu E_\sigma^\nu E_\lambda}\left\{\int_0^{\frac{\pi}{2}} \sin^{2\rho}\theta \cos^{2\sigma}\theta \, I[_\eta^{\xi \sin^{2h}\theta \cos^{2k}\theta}\right] \frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda)c^{\lambda}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda)} \, d\theta\}$$

$$= e^{E_\rho^\mu E_\sigma^\nu E_\lambda}\{\left(\frac{1}{2}\right) I^{0,n_1;m_2,n_2+2;m_3,n_3}_{p_i,q_i:r;p_i'+2,q_i'+1:r';p_i'',q_i'':r''} [_\eta^\xi|^{......,.......:(1/2-\rho,h),(1/2-\sigma,k),......:......,.......}_{......,.......:........,(-\rho-\sigma,h+k):......,.......}] \frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda)c^{\lambda}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda)}\}. \quad (3.4.5)$$

Expanding both sides of (3.4.5) and applying (3.2.3), we have

$$\sum_{r=0}^{\infty}\{\int_0^{\frac{\pi}{2}} \sin^{2(\rho+\mu r)}\theta \cos^{2(\sigma+\nu r)}\theta \, I[_\eta^{\xi \sin^{2h}\theta \cos^{2k}\theta}] \cdot \frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda+r)c^{\lambda+r}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda+r)r!} \, d\theta\}$$

$$=\sum_{r=0}^{\infty}\{\frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda+r)c^{\lambda+r}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda+r)r!}\left(\frac{1}{2}\right) I^{0,n_1;m_2,n_2+2;m_3,n_3}_{p_i,q_i:r;p_i'+2,q_i'+1:r';p_i'',q_i'':r''} [_\eta^\xi|^{......,.......:(1/2-\rho-\mu r,h),(1/2-\sigma-\nu r,k),.....:...,.....}_{......,.......:........,(-\rho-\sigma-(\mu+\nu)r,h+k):......,.......}] \frac{\prod_{j=1}^{u}\Gamma(e_j+\lambda)c^{\lambda}}{\prod_{j=1}^{v}\Gamma(f_j+\lambda)}\}.$$

Further, using $(\alpha, n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ and changing the order of summation and integration on left hand side and then replacing $(e_j + \lambda)$ by e_j and $(f_j + \lambda)$ by f_j, we get (3.4.1).

The results (3.4.2) to (3.4.4) can be derived on the same lines as mentioned above with the help of the results

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta \, I[_\eta^{\xi \sin^{-2h}\theta \cos^{2k}\theta}] \, d\theta$$

$$= (1/2) \, I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+2,q_i'+1:r';p_i'',q_i'':r''} [_\eta^\xi|^{......,.......:(1/2-\sigma,k),......,(1+\rho+\sigma,h-k):......,.......}_{......,.......:(1/2+\rho,h),.........:......,.......}], \quad (2.3.19)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.3.4) and (1.3.5).

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta \, I[_\eta^{\xi \sin^{2h}\theta \cos^{-2k}\theta}] d\theta$$

$$= (1/2) \, I^{0,n_1;m_2+1,n_2+1;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+2:r';p_i'',q_i'':r''} [_\eta^\xi|^{......,.......:(1/2-\rho,h),......:......,.......}_{......,.......:(1/2+\sigma,k),.........,(-\rho-\sigma,h-k):......,.......}], \quad (2.3.20)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.3.4) and (1.3.5).

$$\int_0^{\pi/2} \sin^{2\rho}\theta \cos^{2\sigma}\theta \, I[_\eta^{\xi \sin^{-2h}\theta \cos^{-2k}\theta}] \, d\theta$$

$$= (1/2) \, I^{0,n_1;m_2+2,n_2;m_3,n_3}_{p_i,q_i:r;p_i'+1,q_i'+2:r';p_i'',q_i'':r''} [_\eta^\xi|^{......,.......:........,(1+\rho+\sigma,h+k):......,.......}_{......,.......:(1/2+\rho,h),(1/2+\sigma,k),......:......,.......}], \quad (2.3.21)$$

provided that $\rho > 0$, $\sigma > 0$, $U > 0, V > 0$, $|\arg\xi| < \frac{1}{2}U\pi$, where U and V are given in (1.3.4) and (1.3.5).

respectively.

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