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Existence Theory for Perturbed Abstract Measure Differential Equation

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ABSTRACT

In this paper, an existence results for first abstract measure differential equations is proved under the mixed generalized Lipschitz and Caratheodory's condition.

Keywords: Abstract measure differential equation, fixed point theorems, existence solutions.

INTRODUCTION

1. Statement of Problem

Let IR denote the real line and let IR^n be an $n-$dimensional Euclidean space. We define a norm $|\,.\,|$ in IR^n by

$$|x| = |x_1| + \dots\dots + |x_n|$$

For $x = (x_1, \dots\dots, x_n) \in IR^n$. Let $I_0 = [-r, 0]$ and $I = [\,0, a\,]$ be two closed and bounded intervals in IR . Let $C = C\left(I_0, IR^n\right)$ denote the Banach space of all continuous IR^n-valued functions on I_0 with the usual supremum norm $\|.\|_C$ given by

$$\|\phi\|_C = \sup\left\{ |\phi(\theta)| : -r \le \theta \le 0 \right\}.$$

For any continuous function x defined on the interval where $J = [-r, a] = I_0 \cup I$

Given a function $\phi \in C$, Consider the perturbed Abstract Measure Differential Equation (AMDE)

$$\left.\begin{aligned} \frac{dp}{dx} \in F\left(x, p\left(\bar{S}_x\right)\right) + G\left(x, P\left(\bar{S}_x\right)\right) \quad a.\ e. \quad x \in I \\ p_0 = \phi \end{aligned}\right\} \tag{1.1}$$

Where $F, G : I \times C \to P_f(IR^n)$ and $P_f(IR^n)$ denotes the class of all nonempty subsets of IR^n.

By the solution of AMDE (1.1) we mean a function $\phi \in C(J, IR^n) \cap AC(I, IR^n)$ that satisfies the equation in (1.1), where $AC(J, IR^n)$ is the space of all absolutely continuous functions on J.

The AMDE (1.1) is new and the special cases of it have been discussed in the literature since long time. For example, if $F(x, p(\bar{S}_x)) = \{f(x, p(\bar{S}_x))\}$ and $G(x, p(\bar{S}_x)) = \{g(x, P(\bar{S}_x))\}$, then we obtain a functional differential equation.

$$\left.\begin{aligned} \frac{dp}{dx} &= f(x, p(\bar{S}_x)) + g(x, p(\bar{S}_x)) \ a.e.\ x \in I \\ p_0 &= \phi \end{aligned}\right\} \qquad (1.2)$$

Where $f, g : I \times C \to IR^n$. The more general form of functional differential equation than (1.2) has been discussed in (1.3) for existence results. Again when $G \equiv 0$ on $I \times C$, the AMDE (1.1) reduces to

$$\left.\begin{aligned} \frac{dp}{dx} &\in F(x, P(\bar{S}_x)) \\ p_0 &= \phi \end{aligned}\right\} \qquad (1.3)$$

Where $F : I \times C \to P_f(IR^n)$.

The AMDE (1.3) has already been discussed in the literature via different methods. The multi-valued version of a fixed point theorem of Krasnoselskii is generally used for proving the existence of solution under the mixed Lipshcitzity and Caratheodory's conditions. See Petrusel[10] and the references therein. In this article we shall prove an existence theorem for AMDE (1.1) using a new nonlinear alternative of Schaefer type.

2. AUXILIARY RESULTS

In whole research paper X will be a Banach space and let $P(X)$ denote the class of all subsets of X. Let $P_f(X), P_{bd,cl}(X)$ and $P_{cp,cv}(X)$ and $P_{cp,cv}(X)$ denote respectively the classes of all nonempty, bounded-closed and compact-convex subsets of X. For $x \in X$ and $Y, Z \in P_{bd,cl}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$, and $\|Y\|_p = Sup\{|x| : x \in Y\}$.

Define a function $H : P_{bd,cl}(X) \times P_{bd,cl}(X) \to IR^+$ by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

The function H is called a Hausdorff metric on $P_{bd,cl}(X)$. Note that $\|Y\|_p = H(Y, \{0\})$.

A correspondce $T : X \to P_f(X)$ is called a multivalued mapping on X. A point

$x_0 \in X$ is called a fixed pont of the multivalued operator $T : X \to P_f(X)$ if $x_0 \in T(x_0)$. The fixed points set of T will be denoted by Fix(T).

Definition 2.1: Let $T : X \to P_{bd,\, cl}(X)$ be a mutlivalued operator. Then T is called a multivalued contraction if there exists a constant $K \in (0, 1)$ such that for each $x, y \in X$ we have

$$H\,(T(x), T(y)) \le K\|x - y\|$$

The constant K is called a contraction constant of T.

A multivalued mapping $T : X \to P_f(X)$ is called lower semi-continuous (shortly l.s.c.) (resp.upper semi –continuous(shortly u.s.c.)) if B is any open subset of X then,
$\{ x \in X \mid G_x \cap B \ne \phi \}(resp\{ x \in X \mid G_x \subset B \})$
is an open subset of X .The multivalued operator T is called compact if $\overline{T(X)}$ is a compact subset of X. Again T is called totally bounded if for any bounded subset S of X, A multivalued operator $T : X \to P_f(X)$ is called completely continuous if it is upper semi-continuous and totally bounded on X, for each bounded $A \in P_f(X)$. Every compact multivalued operator is totally bounded but the converse may not true. However the two notions are equivalent on a bounded subset of X.

We apply the following form of the fixed point theorem.

Theorem 2.1: Let X be a Banach space, $A : X \to P_{cl,cv,bd}(X)$ and $B : X \to P_{cp,\,cv}(X)$ two multivalued operators satisfying

(a) A is contraction with a contraction constant k, and

(b) B is completely continuous.

Then either

(i) The operator inclusion $\lambda p \in Ap + Bp$ has a solution for $\lambda = 1$, or

(ii) The set $\varepsilon = \{ u \in X \mid \lambda u \in Au + Bu \ , \lambda > 1 \}$ is unbounded.

We also required definitions which as follow.

Definition 2.2: A multivalued map $F : J \to P_{cp,cv}(IR^n)$ is said to be measurable if for every $y \in IR^n$, the function $d(y, F(t)) = \inf\{\| y - x\| : x \in F(t)\}$ is measurable.

Definition2.3: A multivalued map $F : I \times C \to P_{cl}(IR^n)$ is said to be L^1- Caratheodory if

(i) $x \to F(x, y)$ is measurable for each $y \in C$.

(ii) $y \to F(x, y)$ is upper semi-continuous for almost all $x \in I$, and

(iii) For each real number $\rho > 0$, there exists a function $h_\rho \in L(I, IR^+)$ such that

$$\| F(x,u)\|\rho := \sup\{|v| : v \in F(x,u)\} \le h\rho(x) \ a.e\ x \in J.$$

For all $u \in C$ with $\| u \|_C \leq \rho$.

Denote

$$S_F^1(y) = \left\{ v \in L^1(I, IR^n) : v(x) \in F\left(x, p\left(\bar{S}_x\right)\right) \ a.e. \quad x \in I \right\}$$

Then we have the following lemmas due to Lasota and Opial[9].

Lemma 2.1: If dim $(X) < \infty$ and $F : J \times X \to P_f(X)$ is L^1 – Caratheodory, then $S_F^1(X) \neq \phi$ for each $x \in X$.

Lemma 2.2: Let X be a Banach space, F is an L^1- Caratheodory muti-valued map with $S_F^1 \neq \phi$ and $K : L^1(J, X) \to C(J, X)$ be a linear continuous mapping. Then the operator

$$K_0 S_F^1 : C(J, X) \to P_{cp,cv}(C(J, X))$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

3. EXISTENCE RESULT

We consider the following set of assumptions in the sequel.

(H_1) The multi-function $x \to F(x, y)$ is measurable and integrably bounded for each $y \in C$.

(H_2) There exists a function $k \in L^1(I, IR^+)$ such that the multi-function.

$$F : I \times C \to P_{cl,cv,bd}\left(C\left(I, IR^n\right)\right)$$

Satisfies $H(F(t,x), F(t,y)) \leq k(t) \| x - y \|_C$ $\quad a.e.$ $t \in I$, $\quad$ for all $x, y \in C$ and $\| k \|_{L^1} < 1$.

(H_3) The multi $G(x, y)$ has compact and convex values for each $(x, y) \in I \times C$.

(H_4) G is L^1 – Caratheodory.

(H_5) There exists a function $q \in L^1(I, IR)$ with $q(x) > 0$ for $a.e.$ $x \in I$ and a nondecreaing function $\psi : IR^+ \to (0, \infty)$ such that

$$\| G(x, y) \| \rho := \sup\{ |v| : v \in G(x, y) \} \leq q(x) \psi\left(\| y \|_C\right) a.\ e.\ x \in I \text{ for all } y \in C.$$

Lemma 3.1: Suppose that the assumption (H_2) holds. Then for any $a \in F(x, y)$,

$$| a | \leq k(x) \| y \| c + \| F(x, 0) \| \rho, \quad x \in I \text{ for all } y \in C.$$

Proof: Let $y \in C$ be arbitrary. Then

$$\begin{aligned} \| F(x, y) \| &= H(F(x, y), 0) \\ &\leq H(F(x, y), F(x, 0)) + H(F(x, 0), 0) \\ &\leq H(F(x, y), F(x, 0)) + \| F(x, 0) \| \end{aligned}$$

For all $x \in I$. Hence for any $a \in F(x, y)$

$$\begin{aligned}|a| &\le \|F(x,y)\| \\ &\le H(F(x,y),\, F(x,0)) + \|F(x,0)\|\rho \\ &\le k(x)\|y\|c + \|F(x,0)\|,\end{aligned}$$

For all $x \in I$, hence which complete the proof of the lemma.

Theorem 3.1: Assume that assumptions $(H_1)-(H_5)$ hold. Suppose that

$$\int_{C_0}^{\infty} \frac{ds}{s + \psi(s)} > \|\gamma\|_{L^1} \tag{3.1}$$

Where $c_0 = \int_0^a \|F(s,0)\|\rho\, ds$ and $\gamma(x) = \max\{k(x), q(x)\}$ for $x \in I$. Then the AMDE(1.1) has a solution on J .

Proof: The problem of existence of a solution of AMDE(1.1) reduces to finding the solution of the abstract measure integral equation

$$\left.\begin{aligned} p(x) &\in \phi(0) + \int_0^x F(x, P(\overline{S}_x))\, ds + \int_0^x G(x, P(\overline{S}_x))\, ds, \quad x \in I \\ p(x) &= \phi(x), \qquad x \in I_0 \end{aligned}\right\} \tag{3.2}$$

We study the abtract measure integral equation (3.2) in the space $X = C(I, IR^n)$ of all continuous IR^n –valued function on J with a supremum norm $\|.\|$. Define two multivalued map $A, B: X \to P_f(X)$ by

$$Ap = \left\{ \begin{array}{l} \left\{ u \in C(I, IR^n) : u(x) = \int_0^x v(s)ds,\ v \in S_F^1(y) \right\},\ if\ x \in I \\ 0,\ if\ x \in I_0 \end{array} \right\} \tag{3.3}$$

And

$$Bp = \left\{ \begin{array}{ll} \left\{ u \in C(I, IR^n) : u(x) = \phi(0) + \int_0^x v(s)ds,\ v \in S_G^1(y) \right\}, & if\ x \in I \\ \phi(x), & if\ x \in I_0 \end{array} \right\} \tag{3.4}$$

We shall show that the operators A and B satisfy all the conditions of theorem (2.1) on J .

Step I: First we show that Ap is a closed convex and bounded subset of X for each $p \in X$. This follows easily if we show that the values of Niemytsky operator are closed in $L^1(I, IR^n)$. Let $\{w_n\}$ be a sequence in $L^1(I, IR^n)$ converging to a point w . Then $w_n \to w$ in measure and so, there exists a subset S of positive integers with w_n converging a.e. to w as $n \to \infty$ through

S . Now since assumption (H_1) holds. The values of S_F^1 are closed in $L^1(I, IR^n)$. Thus for each $p \in X$ we have that Ap is non-empty and closed subset of X .

We prove that Ap is a convex subset of X for each $p \in X$. Let $u_1, u_2 \in Ap$. Then there exists v_1 *and* v_2 in $S_F^1(y)$ such that

$$u_j(x) = \int_0^x v_j(s)ds \qquad j = 1, 2.$$

Since $F(x, y)$ has convex values, one has for $0 \le \mu \le 1$,

$$[\mu v_1 + (1-\mu)v_2](x) \in S_F^1(y)(x), \qquad \forall x \in J.$$

We have

$$[\mu u_1 + (1-\mu)u_2](x) = \int_0^x [\mu v_1(s) + (1-\mu)v_2(x)]ds.$$

Therefore $[\mu u_1 + (1-\mu)u_2] \in Ap$ and consequently Ap has convex values in X . From assumption (H_1) it follows that Ap is a bounded subset of X for each $p \in X$. Thus we have $A : X \to P_{cl,cv,bd}(X)$.

Step-II: We show that A is a multivalued contraction on X .

Let $p_1, p_2 \in X$ and $u_1 \in Ap$ then $u_1 \in C(I, IR^n)$ and $u_1(x) = \int_0^x v_1(s)\,ds$ for some $v_1 \in S_F^1(y)$.

Since $H(F(x, p_1(\bar{S}_x)), F(x, P_2(\overline{S_x}))) \le K(x)\|p_1(\bar{S}_x) - p_2(\bar{S}_x)\|_C$, one obtain that there exists $w \in F(x, p_2(\bar{S}_x))$ such that $|v_1(x) - w| \le k(x)\|p_1 - p_2\|_C$. Thus the multivalued operator U defined by $U(x) = S_F^1(p_2)(x) \cap K(x)$,

Where

$$k(x) = \{ w \in IR^n \,|\, |v_1(x) - w| \le k(x)\| p_1 - p_2\|_C \}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U (which exists by Kuratowski-Ryll-Nardzewski's selection theorem. Then $v_2 \in F(x, p_2(\bar{S}_x))$ and, $|v_1(x) - v_2(x)| \le k(x)\|p_1 - p_2\|_C$ a.e. on I .

Define $u_2(x) = \int_o^x v_2(s)ds.$ it follows that $u_2 \in Ap_2$ and

$$|u_1(x) - u_2(x)| \le \left| \int_0^x v_1(s)ds - \int_0^x v_2(s)ds \right|$$

$$\leq \int_0^x |v_1(s) - v_2(s)|\, ds$$

$$\leq \int_0^x k(x) \| p_1(\bar{S}_x) - p_2(\bar{S}x)\|_C\, ds$$

$$\leq \| K \|_{L^1} \|p_1 - p_2\|$$

Taking the supremum over t, we obtain

$$\|u_1 - u_2\| \leq \|k\|_{L^1} \|p_1 - p_2\|.$$

From this and the analogous inequality obtained by interchanging the roles of p_1 and p_2 we get that

$$H(A(p_1), A(p_2)) \leq \|L\|_{L^1} \|p_1 - p_2\|,$$

For all $p_1, p_2 \in X$. This shows that A is a multivalued contraction , since $\|k\|_{L^1} < 1$.

Step III: Now we show that the multivalued operator B is completely continuous on X . First we show that B maps bounded sets into bounded sets in X . To see this, Let Q be a bounded set in X . Then there exists a real number $q > 0$ such that $\|x\|_C \leq q, \forall y \in Q.$

Now for each $u \in Bp$, there exists a $v \in S_G^1(y)$ such that

$$u(x) = \phi(0) + \int_0^x v(s)\, ds$$

Then for each $x \in I$

$$|u(x)| \leq |\phi(0)| + \int_0^x |v(s)|\, ds$$

$$\leq \|\phi\|_C + \int_0^x h_q(s)\, ds$$

$$\leq \|\phi\|_C + \|hq\|_{L^1}$$

This further implies that

$$\|u\| \leq \|\phi\|_C + \|h_q\|_{L^1}$$

For all $u \in Bp \subset \cup B(Q)$. Hence $\cup B(Q)$ is bounded.

Next we show that B maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $u \in Bp$ for some $p \in Q$. Then there exists $v \in S_G^1(y)$ such that

$$u(x) = \phi(0) + \int_0^x v(s)\, ds.$$

Then for any $x_1, x_2 \in I$ with $x_1 \le x_2$ we have

$$|u(x_1) - u(x_2)| \le \left| \int_0^x v(s)ds - \int_0^x v(s)\, ds \right|$$

$$= \int_{x_1}^{x_2} |v(s)|ds$$

$$\le \int_{x_1}^{x_2} h_q(s)ds .$$

If $x_1, x_2 \in I_0$ then $|u(x_1) - u(x_2)| = |\phi(x_1) - \phi(x_2)|$. For the case $x_1 \le 0 \le x_2$ we have

$$|u(x_1) - u(x_2)| \le \left| \phi(x_1) - \phi(0) - \int_0^{x_2} v(s)ds \right|$$

$$\le |\phi(x_1) - \phi(0)| + \int_0^x |v(s)|$$

$$\le |\phi(x_1) - \phi(0)| + \int_0^{x_2} h_r(s)ds.$$

Hence, in all cases, we have

$$|u(x_1) - u(x_2)| \to 0 \quad as \ x_1 \to x_2 .$$

As a result $\cup B(Q)$ is an equicontinuous set in X. Now an application of Arzela- Ascoli theorem yields that that the multi-valued operator B is totally bounded on X.

Step IV: Next we prove that B has a closed graph.
Let $\{x_n\} \subset X$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bp_n$ for each $n \in N$ such that $y_n \to y_*$. We will show that $y_* \in Bp_*$. Since $y_n \in Bp_{n.}$. There exists a $v_n \in S_G^1(x_n)$ such that

$$y_n(x) = \phi(0) + \int_0^x v_n(s)ds .$$

Consider the linear and continuous operator $K : L^1(J, IR^n) \to X$ defined by

$$Kv(x) = \int_0^x v_n(s)ds$$

Now,

$$|y_n(x) - \phi(0) - (y_*(x) - \phi(0))| \le |y_n(x) - y_*(x)|$$

$$\leq \| y_n - y_* \| \to 0 \text{ as } n \to \infty$$

From Lemma (2.2) It follows that $(K_0 S_G^1)$ is a closed graph operator and from the definition of K one has

$$y_n(x) - \phi(0) \in (K_0 S_F^1(x_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v \in S_G^1(x_*)$ such that

$$y_*(x) = \phi(0) + \int_0^x v_*(s) ds.$$

Hence the multivalued operator B is an upper semi-continuous operator on X.

Step VI: Finally we show that the set

$$\varepsilon = \{u \in X : \lambda u \in Au + Bu \quad for\ some\ \lambda > 1\} \text{ is bounded.}$$

Let $u \in \varepsilon$ be any element. Then there exists $v_1 \in S_F^1(u)$ and $v_2 \in S_G^1(u)$ such that

$$u(x) = \lambda^{-1} \phi(0) + \lambda^{-1} \int_0^x v_1(s) ds + \lambda^{-1} \int_0^x v_2(s) ds.$$

Then

$$|u(x)| \leq \phi(0) + \int_0^x |v_1(s)| ds + \int_0^x |v_2(s)| ds$$

$$\leq \phi(0) + \int_0^x \left(k(s) \| u_s \|c + \| F(s, o) \| \right) ds + \int_0^x q(s) \psi(\| u_s \|c) ds.$$

Put $w(x) = \max\{|u(s)| : -r \leq s \leq x\}, x \in I$. Then $\| ux \|c \leq w(x)$ for all $x \in I$ and there is a point $x \in I$ and there is a point $x^* \in [-r, x]$ such that $w(x) = u(x^*)$. Hence we have

$$w(x) = |u(x^*)|$$

$$\leq \int_0^{x^*} k(s) \| u_s \|c \, ds + \int_0^{x^*} \| F(s, 0) \| ds + \int_0^{x^*} q(s) \psi(\|u_s\| c) ds$$

$$\leq c_0 + \int_0^x k(s) w(s) ds + \int_0^x q(s) \psi(w(s)) ds$$

$$\leq c_0 + \int_0^x \gamma(s) (w(s) + \psi(w(s))) ds.$$

Let

$$m(x) = c_0 + \int_0^x \gamma(s) (w(s) + \psi(w(s))) ds, \quad x \in I$$

Then we have $w(x) \leq m(x)$ for all $x \in I$. Differentiating the both sides of the above equation, we obtain

$$\frac{dm}{dx} = \gamma(x)(w(x) + \psi(w(x))), \quad a.e. \quad x \in I, \; m(0) = c_0.$$

This further implies that

$$\frac{dm}{dx} \leq \gamma(x)(m(x) + \psi(m(x))) \; a.e. \; x \in I, \; m(0) = c_0,$$

That is,

$$\frac{\frac{dm}{dx}}{m(x) + \psi(m(x))} \leq \gamma(x) \; a.e. \quad x \in J, \; m(0) = c_0.$$

Integrating from 0 *to* t, we get

$$\int_0^x \frac{\frac{dm}{ds}}{m(s) + \psi(m(s))} ds \leq \int_0^x \gamma(s) ds.$$

By the change of variable.

$$\int_{c_0}^{m(x)} \frac{ds}{s + \psi(s)} \leq \| \gamma \|_{L^1} < \int_{c_0}^{\infty} \frac{1}{s + \psi(s)} ds$$

Hence there exists a constant M such that

$$w(x) \leq m(x) \leq M \text{ for all } x \in I.$$

Now from the definition of w it follows that

$$\| u \| = \sup_{x \in [-r, a]} |u(x)| = w(a) \leq m(a) \leq M,$$

For all $u \in \varepsilon$. This shows that the set ε is bounded in X. As a result the conclusion (ii) of theorem (2.1) does not hold. Hence the conclusion (i) holds and consequently (3.2) or equivalent AMDE (1.1) has a solution p on J. Hence the proof is completed.

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