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A Bulk Arrival Retrial Queue with Feedback and Exponentially Distributed Multiple Working Vacation

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ABSTRACT

We consider a bulk arrival retrial queue with feedback and exponentially distributed multiple working vacation. The server provides service to customers, one by one, on a FCFS basis. Just after completion of his service a customer may leave the system or may opt to repeat his service, in which case this customer rejoins the head of the queue. After completion of customer's service there is no customer in the orbit the server may take a multiple working vacation. Using supplementary variable method we obtain the probability generating function for the number of customers in the orbit. Some particular cases are discussed.

MSC: 60K25, 60K30.

Keywords: Bulk arrival retrial queues, Feedback, Linear retrial policy, Working vacation and Supplementary variable method.

1. INTRODUCTION

Retrial queueing systems are described by the feature that the arriving customers who find the server busy join the retrial orbit to try their requests agian. Retrial queues are widely and successfully used as Mathematical models of several Computer systems and telecommunication networks.

Choi *et al.*[4] analysed an $M/M/1$ retrial queue with general retrial times. Martin and Gomez-Corral[12] considered an $M/G/1$ retrial queue with linear control policy, Lillo[11] investigated a $G/M/1$ retrial queue. Sherman and Kharoufeh[17] studied an $M/M/1$ retrial queue with unreliable server.

One additional feature which has been widely discussed in retrial queueing system is the Bernoulli feedback of customers. The phenomena of feedback in retrial queueing systems occured in many practical situations. For example, the retrial queue with feedback can be used to model the Automative Repeat Request protocal in a high frequency communication network. Falin[16] studied an $M/M/1$ retrial queue with feedback. Kumar *et al*.[8] investigated an $M/G/1$ retrial queue with feedback and starting failures. Ke and Chang[6] considered a modified vacation policy for the $M/G/1$ retrial queue with balking and feedback. Kumar *et al*.[9] discussed an $M/M/1$ retrial queue with feedback and collisions. Kumar *et al*.[10] analyzed an $M/G/1$ retrial queue with feedback and negative customers.

In the queueing theory, vacation queues and retrial queues have been intensive research topics; we can find general models in Artalejo[2] and Gomez-Corral[12]. The study of queueing system with working vacations can also provide the theory and analysis method to design the optimal lower speed period.In 2002, Servi and Finn[16] first introduced working vacation policy and studied an $M/M/1/WV$ queue.

Wu and Takagi[18] extended the $M/M/1/WV$ queue to an $M/G/1/WV$ queue using the matrix-analytic method, Baba[5] considered a $GI/M/1$ queue with working vacations. Krishnamoorthy and Sreenivasan[7] analysed an $M/M/2$ queue with working vacations.

Do[5] studied an $M/M/1$ retrial queue with working vacation. Zhang and Xu[19] considered an $M/M/1/WV$ queue with N-policy. Aftab Begum[1] studied $M^X/G/1$ queue with exponentially distributed Multiple working vacations and Santhi and Pazhani Bala Murugan[14] analysed $M^X/G/1$ queue with Multiple working vacation and with feedback. Pazhani Bala Murugan and Vijaykrishnaraj[15] studied A bulk arrival retrial queue with exponentially distributed multiple working vacation.

In this paper, we study a bulk arrival retrial queueing system with feedback and exponentially distributed multiple working vacation. The organization of the paper is as follows. In section 2, we described the model. In section 3, we obtained the steady state probability generating function. In section 4 some particular cases have been discussed.

2. MODEL DESCRIPTION

We consider an $M^X/G/1$ queueing system where the primary customers arrive according to a compound Poisson process with arrival rate $\lambda(\geq 0)$. The batch size X is a random variable $Pr(X = n) = g_n, n = 1,2,3, \ldots$ with probability generating function $X(z) = \sum_{k=1}^{\infty} g_k z^k$ and the first and second factorial moments of X are defined by $X^{(1)}(1) = E(X)$ and $X^{(2)}(1) = E(X(X - 1))$.

We assume that there is no waiting space and therefore if an arriving customer (external or repeated) finds the server occupied, he leaves the service area and joins a pool of blocked customers called orbit. We will assume that only the customer at the head of the orbit is allowed to reach the server at a service completion instant. The retrial time follows a general distribution, with distibution functions $B(x)$. Let $b(x)$ and $B^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $B(x)$ respectively for regular service period and let $a(x), A^*(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $A(x)$ respectively for working vacation period. Just after the completion of a service, if any customer is in orbit the next customer to gain service is determined by a competition between the primary customer and the orbit customer.

The service time is assumed to follow general distribution, with distribution function $S_b(x)$ and density funciton $s_b(x)$. Let $S_b^*(\theta)$ be the Laplace Stieltjes Transform (LST) of the service time S_b.

Whenever the orbit becomes empty at a service completion instant the server starts a working vacation and the duration of the vacation time follows an exponential distribution with rate η. At a vacation completion instant if there are customers in the system the server will start a new busy period. Otherwise he takes another working vacation. This type of vacation policy is called multiple working vacation. During the working vacation period, the server provides service with service time S_v which follows a general distribution with distribution function $S_v(x)$. Let$s_v(x)$ be the probability density function and $S_v^*(\theta)$ be the Laplace Stieltjes Transform of $S_v(x)$. Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with different distribution at the beginning of the following service period.

After completion of each service a customer may like to repeat his service with probability p or may leave the system with probability $q = 1 - p$ in both not working vacation period and working vacation period.

We assume that inter-arrival times, service times, working vacation times and a retrial times are mutually independent.

We define the following random variables

N(t)-the orbit size at time t.

$A^0(t)$-the remaining retrial time in working vacation period.

$B^0(t)$-the remaining retrial time in regular service period.

$S_v^0(t)$-the remaining service time in working vacation period.

$S_b^0(t)$-the remaining service time in regular service period.

$$Y(t) = \begin{cases} 0 \text{ if the server is on working vacation period at time t but not occupied} \\ 1 \text{ if the server is in regular service period at time t but not occupied} \\ 2 \text{ if the server is busy on working vacation period at time t} \\ 3 \text{ if the server is busy in regular service period at time t} \end{cases}$$

so that the supplementary variables $A^0(t), B^0(t), S_v^0(t)$ and $S_b^0(t)$ are introduced in order to obtain the bivaritate Markov Process $\{N(t), \partial(t); t \geq 0\}$, Where

$$\partial(t) = \begin{cases} A^0(t) \text{ if } Y(t) = 0 \\ B^0(t) \text{ if } Y(t) = 1 \\ S_v^0(t) \text{ if } Y(t) = 2 \\ S_b^0(t) \text{ if } Y(t) = 3 \end{cases}$$

We define the following limiting probabilites:

$Q_{0,0} = \lim_{t\to\infty} Pr\{N(t) = 0, Y(t) = 0\}$

$Q_{0,n} = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 0, x < A^0(t) \le x + dx\}; n \ge 1$

$P_{0,n} = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 1, x < B^0(t) \le x + dx\}; n \ge 1$

$Q_{1,n} = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 2, x < S_v^0(t) \le x + dx\}; n \ge 0$

$P_{1,n} = \lim_{t\to\infty} Pr\{N(t) = n, Y(t) = 3, x < S_b^0(t) \le x + dx\}; n \ge 0$

We define the Laplace Stieltjes Transform and the probability generating functions as follows,

$S_b^*(\theta) = \int_0^\infty e^{-\theta x} s_b(x)dx;$ $\quad S_v^*(\theta) = \int_0^\infty e^{-\theta x} s_v(x)dx;$ $\quad A^*(\theta) = \int_0^\infty e^{-\theta x} a(x)dx$

$B^*(\theta) = \int_0^\infty e^{-\theta x} b(x)dx;$ $\quad Q_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{0,n}(x)dx;$ $\quad Q_{0,n}^*(0) = \int_0^\infty Q_{0,n}(x)dx;$

$Q_{1,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{1,n}(x)dx;$ $\quad Q_{1,n}^*(0) = \int_0^\infty Q_{1,n}(x)dx,$ $\quad P_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} P_{0,n}(x)dx;$

$P_{0,n}^*(0) = \int_0^\infty P_{0,n}(x)dx;$ $\quad Q_0^*(z,\theta) = \sum_{n=1}^\infty Q_{0,n}^*(\theta)z^n;$ $\quad Q_0^*(z,0) = \sum_{n=1}^\infty Q_{0,n}^*(0)z^n;$

$Q_0(z,0) = \sum_{n=1}^\infty Q_{0,n}(0)z^n;$ $\quad Q_1^*(z,\theta) = \sum_{n=0}^\infty Q_{1,n}^*(\theta)z^n;$ $\quad Q_1^*(z,0) = \sum_{n=0}^\infty Q_{1,n}^*(0)z^n;$

$Q_1(z,0) = \sum_{n=0}^\infty Q_{1,n}(0)z^n;$ $\quad P_0^*(z,\theta) = \sum_{n=1}^\infty P_{0,n}^*(\theta)z^n;$ $\quad P_0^*(z,0) = \sum_{n=1}^\infty P_{0,n}^*(0)z^n;$

$P_0(z,0) = \sum_{n=1}^\infty P_{0,n}(0)z^n;$ $\quad P_1^*(z,\theta) = \sum_{n=0}^\infty P_{1,n}^*(\theta)z^n;$ $\quad P_1^*(z,0) = \sum_{n=0}^\infty P_{1,n}^*(0)z^n;$

$P_1(z,0) = \sum_{n=0}^\infty P_{1,n}(0)z^n.$

3. THE ORBIT SIZE DISTRIBUTION

By assuming that the system is in steady state condition, the differential difference equations governing the systems are as follows:

$$\lambda Q_{0,0} = qP_{1,0}(0) + qQ_{1,0}(0) \quad (1)$$

$$-\frac{d}{dx}Q_{0,n}(x) = -(\lambda+\eta)Q_{0,n}(x) + qQ_{1,n}(0)a(x); n \ge 1 \quad (2)$$

$$-\frac{d}{dx}Q_{1,0}(x) = -(\lambda+\eta)Q_{1,0}(x) + Q_{0,1}(0)s_v(x) + \lambda Q_{0,0}s_v(x)g_1 + pQ_{1,0}(0)s_v(x) \quad (3)$$

$$-\frac{d}{dx}Q_{1,n}(x) = -(\lambda+\eta)Q_{1,n}(x) + \sum_{k=1}^{n} \lambda Q_{1,n-k}(x)g_k + Q_{0,n+1}(0)s_v(x)$$
$$+\lambda s_v(x)\int_0^\infty \sum_{k=1}^n Q_{0,n-k+1}(x)g_k dx + \lambda g_{n+1}Q_{0,0}s_v(x) + pQ_{1,n}(0)s_v(x) \quad (4)$$

$$-\frac{d}{dx}P_{0,n}(x) = -\lambda P_{0,n}(x) + qP_{1,n}(0)b(x) + \eta b(x)\int_0^\infty Q_{0,n}(x)dx \tag{5}$$

$$-\frac{d}{dx}P_{1,0}(x) = -\lambda P_{1,0}(x) + P_{0,1}(0)s_b(x) + pP_{1,0}(0)s_b(x) + \eta s_b(x)\int_0^\infty Q_{1,0}(x)dx \tag{6}$$

$$-\frac{d}{dx}P_{1,n}(x) = -\lambda P_{1,n}(x) + \sum_{k=1}^{n}\lambda P_{1,n-k}(x)g_k + P_{0,n+1}(0)s_b(x) + pP_{1,n}(0)s_b(x)$$
$$+\lambda s_b(x)\int_0^\infty \sum_{k=1}^{n} P_{0,n-k+1}(x)g_k dx + \eta s_b(x)\int_0^\infty Q_{1,n}(x)dx \tag{7}$$

Taking LST on both sides of the equation from (2) to (7) we get,

$$\theta Q^*_{0,n}(\theta) - Q_{0,n}(0) = (\lambda+\eta)Q^*_{0,n}(\theta) - qQ_{1,n}(0)A^*(\theta); n \geq 1 \tag{8}$$

$$\theta Q^*_{1,0}(\theta) - Q_{1,0}(0) = (\lambda+\eta)Q^*_{1,0}(\theta) - Q_{0,1}(0)S^*_v(\theta) - \lambda g_1 Q_{0,0}S^*_v(\theta) - pQ_{1,0}(0)S^*_v(\theta) \tag{9}$$

$$\theta Q^*_{1,n}(\theta) - Q_{1,n}(0) = (\lambda+\eta)Q^*_{1,n}(\theta) - \sum_{k=1}^{n}\lambda g_k Q^*_{1,n-k}(\theta) - Q_{0,n+1}(0)S^*_v(\theta)$$
$$-\lambda g_{n+1}Q_{0,0}S^*_v(\theta) - \lambda\sum_{k=1}^{n} Q^*_{0,n+1}(0)g_k S^*_v(\theta) - pQ_{1,0}(0)S^*_v(\theta) \tag{10}$$

$$\theta P^*_{0,n}(\theta) - P_{0,n}(0) = \lambda P^*_{0,n}(\theta) - qP_{1,n}(0)B^*(\theta) - \eta Q^*_{0,n}(0)B^*(\theta) \tag{11}$$

$$\theta P^*_{1,0}(\theta) - P_{1,0}(0) = \lambda P^*_{1,0}(\theta) - P_{0,1}(0)S^*_b(\theta) - pP_{1,0}(0)S^*_b(\theta) - \eta Q^*_{1,0}S^*_b(\theta) \tag{12}$$

$$\theta P^*_{1,n}(\theta) - P_{1,n}(0) = \lambda P^*_{1,n}(\theta) - \sum_{k=1}^{n}\lambda g_k P^*_{1,n-k}(\theta) - P_{0,n+1}(0)S^*_b(\theta) - pP_{1,n}(0)S^*_b(\theta)$$
$$-\lambda S^*_b(\theta)\sum_{k=1}^{n} P^*_{0,n-k+1}(0)g_k - \eta S^*_b(\theta)Q^*_{1,n}(0) \tag{13}$$

Multiplying (8) with z^n and summed over n from 1 to ∞, we get

$$\theta\sum_{n=1}^{\infty} Q^*_{0,n}(\theta)z^n - \sum_{n=1}^{\infty} Q_{0,n}(0)z^n = (\lambda+\eta)\sum_{n=1}^{\infty} Q^*_{0,n}(\theta)z^n - qA^*(\theta)\sum_{n=1}^{\infty} Q_{1,n}(\theta)z^n$$

$$[\theta-(\lambda+\eta)]Q^*_0(z,\theta) = Q_0(z,0) - qA^*(\theta)Q_1(z,0) + qA^*(\theta)Q_{1,0}(0) \tag{14}$$

z^n times (10) summed over n from 1 to ∞ and added up with (9) gives

$$\theta\sum_{n=0}^{\infty} Q^*_{1,n}(\theta)z^n - \sum_{n=0}^{\infty} Q_{1,n}(\theta)z^n = (\lambda+\eta)\sum_{n=0}^{\infty} Q^*_{1,n}(\theta)z^n - \lambda\sum_{n=1}^{\infty}\sum_{k=1}^{n} g_k Q^*_{1,n-k}(\theta)z^n$$
$$-S^*_v(\theta)\sum_{n=0}^{\infty} Q_{0,n+1}(0)z^n - \lambda Q_{0,0}S^*_v(\theta)\sum_{n=0}^{\infty} g_{n+1}z^n$$
$$-pS^*_v(\theta)\sum_{n=0}^{\infty} Q_{1,n}(0)z^n - \lambda S^*_v(\theta)\sum_{n=1}^{\infty}\sum_{k=1}^{n} Q^*_{0,n+1-k}(0)g_k z^n$$

$$[\theta-(\lambda-\lambda X(z)+\eta)]Q^*_1(z,\theta) = [1-pS^*_v(\theta)]Q_1(z,0) - \frac{S^*_v(\theta)}{z}Q_0(z,0) - \lambda Q_{0,0}X(z)\frac{S^*_v(\theta)}{z}$$
$$-\frac{\lambda S^*_v(\theta)}{z}X(z)Q^*_0(z,0) \tag{15}$$

Inserting $\theta = \lambda+\eta$ in (14), we get

$$Q_0(z,0) = qA^*(\lambda+\eta)[Q_1(z,0) - Q_{1,0}(0)] \tag{16}$$

Substituting $\theta = 0$ in (14) and using (16), we get

$$Q_0^*(z,0) = \frac{q[1-A^*(\lambda+\eta)][Q_1(z,0)-Q_{1,0}(0)]}{\lambda+\eta} \tag{17}$$

Inserting $\theta = \lambda - \lambda X(z) + \eta$ in (15), we get

$$Q_1(z,0) = \frac{S_v^*(\lambda-\lambda X(z)+\eta)[-q[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]Q_{1,0}(0)+\lambda(\lambda+\eta)X(z)Q_{0,0}]}{(\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)]-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]} \tag{18}$$

Substituting (18) in (16), we get

$$Q_0(z,0) = qA^*(\lambda+\eta)[\frac{S_v^*(\lambda-\lambda X(z)+\eta)\lambda(\lambda+\eta)X(z)Q_{0,0}-(\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)]Q_{1,0}(0)}{(\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)]-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]} \tag{19}$$

Let as consider

$$f(z) = (\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)] - qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z) + (\lambda-\lambda X(z) +\eta)A^*(\lambda+\eta)],$$

we find $f(0) < 0$ and $f(1) > 0$.
This implies that there exist a real root $z_1 \in (0,1)$ for the equation $f(z) = 0$.
Hence at $z = z_1$ the equation (19) becomes

$$Q_{1,0}(0) = \frac{\lambda X(z_1)S_v^*(\lambda-\lambda X(z_1)+\eta)Q_{0,0}}{z_1[1-pS_v^*(\lambda-\lambda X(z_1)+\eta)]} \tag{20}$$

Substituting (20) in (18), we get

$$Q_1(z,0) = \frac{S_v^*(\lambda-\lambda X(z)+\eta)[\lambda(\lambda+\eta)X(z)-\lambda U(z_1)q[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]Q_{0,0}}{(\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)]-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]} \tag{21}$$

Substituting (20) in (19), we get

$$Q_0(z,0) = \frac{q\lambda(\lambda+\eta)A^*(\lambda+\eta)[X(z)S_v^*(\lambda-\lambda X(z)+\eta)-U(z_1)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z]Q_{0,0}}{(\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)]-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]} \tag{22}$$

Where $U(z) = \frac{X(z)}{z}S_v^*(\lambda-\lambda X(z)+\eta)$

Substituting (20) and (21) in (17), we get

$$Q_0^*(z,0) = \frac{\lambda q[1-A^*(\lambda+\eta)][X(z)S_v^*(\lambda-\lambda X(z)+\eta)-U(z_1)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z]Q_{0,0}}{(\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)]-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]} \tag{23}$$

Inserting $\theta = 0$ and substituting (21), (22) and (23) in (15), we get

$$Q_1^*(z,0) = \frac{[1-S_v^*(\lambda-\lambda X(z)+\eta)]\left[(\lambda+\eta)X(z)-qU(z_1)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]\right]\lambda Q_{0,0}}{(\lambda-\lambda X(z)+\eta)\left[(\lambda+\eta)z[1-pS_v^*(\lambda-\lambda X(z)+\eta)]-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]\right]} \tag{24}$$

Multiplying (11) with z^n and summed over n from 1 to ∞, we get

$$\theta\sum_{n=1}^{\infty} P_{0,n}^*(\theta)z^n - \sum_{n=1}^{\infty} P_{0,n}(0)z^n = \lambda\sum_{n=1}^{\infty} P_{0,n}^*(\theta)z^n - qB^*(\theta)\sum_{n=1}^{\infty} P_{1,n}(0)z^n - \eta B^*(\theta)\sum_{n=1}^{\infty} Q_{0,n}^*(0)z^n$$

$$(\theta-\lambda)P_0^*(z,\theta) = P_0(z,0) - qB^*(\theta)P_1(z,0) + B^*(\theta)qP_{1,0}(0) - \eta B^*(\theta)Q_0^*(z,0) \tag{25}$$

z^n times (13) is summed over n from 1 to ∞ and added up with (12) gives

$$\theta\sum_{n=0}^{\infty} P_{1,n}^{*}(\theta)z^{n}-\sum_{n=0}^{\infty} P_{1,n}(0)z^{n} \quad = \quad \lambda\sum_{n=0}^{\infty} P_{1,n}^{*}(\theta)z^{n}-\lambda\sum_{n=1}^{\infty}\sum_{k=1}^{n} g_{k}P_{1,n-k}^{*}(\theta)z^{n}-S_{b}^{*}(\theta)\sum_{n=0}^{\infty} P_{0,n+1}(0)z^{n}$$
$$-pS_{b}^{*}(\theta)\sum_{n=0}^{\infty} P_{1,n}(0)z^{n}-\lambda S_{b}^{*}(\theta)\sum_{n=1}^{\infty}\sum_{k=1}^{n} P_{0,n-k+1}^{*}(0)g_{k}z^{n}$$
$$-\eta S_{b}^{*}(\theta)\sum_{n=0}^{\infty} Q_{1,n}^{*}(0)z^{n}$$

$$\left[\theta-\left(\lambda-\lambda X(z)\right)\right]P_{1}^{*}(z,\theta)=[1-pS_{b}^{*}(\theta)]P_{1}(z,0)-\frac{S_{b}^{*}(\theta)}{z}P_{0}(z,0)-\frac{\lambda X(z)}{z}S_{b}^{*}(\theta)P_{0}^{*}(z,0)$$
$$-\eta S_{b}^{*}(\theta)Q_{1}^{*}(z,0) \quad (26)$$

Inserting $\theta=\lambda$ and substituting $qP_{1,0}(0)=[1-qU(z_{1})]\lambda Q_{0,0}$ in (25),we get

$$P_{0}(z,0)=qB^{*}(\lambda)P_{1}(z,0)-B^{*}(\lambda)[1-qU(z_{1})]\lambda Q_{0,0}+\eta B^{*}(\lambda)Q_{0}^{*}(z,0) \quad (27)$$

Inserting $\theta=\lambda-\lambda X(z)$ and substituting (27) in (26), we get

$$[1-pS_{b}^{*}(\lambda-\lambda X(z))]P_{1}(z,0)=\frac{S_{b}^{*}\left(\lambda-\lambda X(z)\right)}{z}[qB^{*}(\lambda)P_{1}(z,0)-B^{*}(\lambda)[1-qU(z_{1})]\lambda Q_{0,0}$$
$$+\eta B^{*}(\lambda)Q_{0}^{*}(z,0)]+\frac{\lambda X(z)S_{b}^{*}\left(\lambda-\lambda X(z)\right)}{z}P_{0}^{*}(z,0)$$
$$+\eta S_{b}^{*}(\lambda-\lambda X(z))Q_{1}^{*}(z,0) \quad (28)$$

Inserting $\theta=0$ and substituting (27) and $qP_{1,0}(0)=[1-U(z_{1})]\lambda Q_{0,0}$ in (25), we get

$$P_{0}^{*}(z,0)=\frac{[1-B^{*}(\lambda)]}{\lambda}[qP_{1}(z,0)-[1-qU(z_{1})]\lambda Q_{0,0}+\eta Q_{0}^{*}(z,0)] \quad (29)$$

Substituting (29) and (27)in (28), we get

$$P_{1}(z,0)=-\frac{[1-qU(z_{1})][B^{*}(\lambda)+X(z)(1-B^{*}(\lambda))]S_{b}^{*}(\lambda-\lambda X(z))\lambda Q_{0,0}+\eta BS_{b}^{*}(\lambda-\lambda X(z))Q_{0}^{*}(z,0)+\eta zS_{b}^{*}(\lambda-\lambda X(z))Q_{1}^{*}(z,0)}{[1-pS_{b}^{*}(\lambda-\lambda X(z))]z-qS_{b}^{*}(\lambda-\lambda X(z))[B^{*}(\lambda)+X(z)(1-B^{*}(\lambda))]} \quad (30)$$

Substituting (30) in (27), we get

$$P_{0}(z,0)=\frac{[1-pS_{b}^{*}(\lambda-\lambda X(z))]zB^{*}(\lambda)[\eta Q_{0}^{*}(z,0)-[1-qU(z_{1}]\lambda Q_{0,0}]+B^{*}(\lambda)\eta z\mathrm{Q}_{1}^{*}(z,0)qS_{b}^{*}(\lambda-\lambda X(z))}{[1-pS_{b}^{*}(\lambda-\lambda X(z))]z-qS_{b}^{*}(\lambda-\lambda X(z))[B^{*}(\lambda)+(1-B^{*}(\lambda))X(z)]} \quad (31)$$

Substituting (30) in (29), we get

$$P_{0}^{*}(z,0)=[\frac{1-B^{*}(\lambda)}{\lambda}][\frac{\eta zqS_{b}^{*}(\lambda-\lambda X(z))Q_{1}^{*}(z,0)+[1-pS_{b}^{*}(\lambda-\lambda X(z))]z\eta Q_{0}^{*}(z,0)-[1-pS_{b}^{*}(\lambda-\lambda X(z))]z[1-qU(z_{1})]\lambda Q_{0,0}}{[1-pS_{b}^{*}(\lambda-\lambda X(z))]z-qS_{b}^{*}(\lambda-\lambda X(z))[B^{*}(\lambda)+(1-B^{*}(\lambda)X(z))]}] \quad (32)$$

Substituting (23) and (24)in (32), we get

$$P_{0}^{*}(z,0)=\frac{N_{1}(z)}{D_{1}(z)}.Q_{0,0} \quad (33)$$

Where,

$$N_{1}(z)=[1-B^{*}(\lambda)]\eta zqS_{b}^{*}(\lambda-\lambda X(z))[1-S_{v}^{*}(\lambda-\lambda X(z)+\eta)][(\lambda+\eta)X(z)$$
$$-qU(z_{1})[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^{*}(\lambda+\eta)]]+[1-B^{*}(\lambda)][\lambda-\lambda X(z)$$
$$+\eta][1-pS_{b}^{*}(\lambda-\lambda X(z))]\eta zq[1-A^{*}(\lambda+\eta)][X(z)S_{v}^{*}(\lambda-\lambda X(z)+\eta)$$
$$-U(z_{1})[1-pS_{v}^{*}(\lambda-\lambda X(z)+\eta)]z]-[1-B^{*}(\lambda)][\lambda-\lambda X(z)+\eta][1$$
$$-pS_{b}^{*}(\lambda-\lambda X(z))]z[1-qU(z_{1})][(\lambda+\eta)[1-pS_{v}^{*}(\lambda-\lambda X(z)+\eta]z$$
$$-qS_{v}^{*}(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^{*}(\lambda+\eta)]]$$

$$D_{1}(z)=[[1-pS_{b}^{*}(\lambda-\lambda X(z))]z-qS_{b}^{*}(\lambda-\lambda X(z))[B^{*}(\lambda)+(1-B^{*}(\lambda))X(z)]][\lambda$$
$$-\lambda X(z)$$

$$+\eta][(\lambda+\eta)[1-pS_v^*(\lambda-\lambda X(z)+\eta]z-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$$

Substituting $\theta = 0$ and Substituting (24), (27), (30) and (33) in (26) we get

$$P_1^*(z,0) = \frac{N_2(z)}{D_2(z)}.\lambda Q_{0,0} \tag{34}$$

Where,

$$N_2(z) = \eta z[1-S_b^*(\lambda-\lambda X(z)][1-S_v^*(\lambda-\lambda X(z)+\eta)][(\lambda+\eta)X(z)-qU(z_1)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]+\eta q[B^*(\lambda)+(1-B^*(\lambda)X(z)][1-S_b^*(\lambda-\lambda X(z)][\lambda-\lambda X(z)+\eta][1-A^*(\lambda+\eta)][X(z)S_v^*(\lambda-\lambda X(z)+\eta)-U(z_1)[1-pS_v^*(\lambda-\lambda X(z)+\eta]z]$$

$$D_2(z) = [\lambda-\lambda X(z)][\lambda-\lambda X(z)+\eta][[1-pS_b^*(\lambda-\lambda X(z))]z-qS_b^*(\lambda-\lambda X(z))[B^*(\lambda)+(1-B^*(\lambda)X(z)]][(\lambda+\eta)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$$

We define $P_V(z) = Q_0^*(z,0)+Q_1^*(z,0)+Q_{(0,0)}$

$$P_V(z) = \frac{N_V(z)}{D_V(z)}.Q_{0,0} \tag{35}$$

as the probability generating function for the number of customers in the orbit when the server is on working vacation period.

Where,

$$N_V(z) = \lambda q[1-A^*(\lambda+\eta)][\lambda-\lambda X(z)+\eta)][X(z)S_v^*(\lambda-\lambda X(z)+\eta)-U(z_1)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z]+\lambda[1-S_v^*(\lambda-\lambda X(z)+\eta)][(\lambda+\eta)X(z)-qU(z_1)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]+[\lambda-\lambda X(z)+\eta)][(\lambda+\eta)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]$$

$$D_V(z) = [\lambda-\lambda X(z)+\eta][(\lambda+\eta)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]]$$

and

$$P_B(z) = P_0^*(z,0)+P_1^*(z,0)$$

$$P_B(z) = \frac{N_B(z)}{D_B(z)}Q_{0,0} \tag{36}$$

as the probability generating function for the number of customers in the orbit when the server is on not working vacation (normal busy) period.

Where,

$$N_B(z) = \lambda\eta z[1-S_v^*(\lambda-\lambda X(z)+\eta)][(\lambda+\eta)X(z)-qU(z_1)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)]][[1-pS_b^*(\lambda-\lambda X(z)]-qS_b^*(\lambda-\lambda X(z))[B^*(\lambda)+(1-B^*(\lambda))X(z)]]+\lambda\eta q[1-A^*(\lambda+\eta)][\lambda-\lambda X(z)+\eta][X(z)S_v^*(\lambda-\lambda X(z)+\eta)-U(z_1)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z][z[1-pS_b^*(\lambda-\lambda X(z))]+[1-z-S_b^*(\lambda-\lambda X(z))+zpS_b^*(\lambda-\lambda X(z))][B^*(\lambda)+(1-B^*(\lambda)X(z)]]-\lambda[1-qU(z_1)][\lambda-\lambda X(z)+\eta][(\lambda+\eta)[1-pS_v^*(\lambda-\lambda X(z)+\eta)]z-qS_v^*(\lambda-\lambda X(z)+\eta)[\lambda X(z)+(\lambda-\lambda X(z)+\eta)A^*(\lambda+\eta)][z[1-pS_b^*(\lambda-\lambda X(z))]+[1-z-S_b^*(\lambda-\lambda X(z))+zpS_b^*(\lambda-\lambda X(z))][B^*(\lambda)+(1-B^*(\lambda))X(z)]]$$

$$D_B(z) = [\lambda - \lambda X(z)][\lambda - \lambda X(z) + \eta][[1 - pS_b^*(\lambda - \lambda X(z)]z - qS_b^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda)X(z)]][(\lambda + \eta)[1 - pS_v^*(\lambda - \lambda X(z) + \eta)]z - qS_v^*(\lambda - \lambda X(z) + \eta)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]]$$

We define

$$P(z) = P_B(z) + P_V(z) \tag{37}$$

as the probability generating function for the number of customers in the orbit irrespective of the server state.

Where $P_V(z)$ and $P_B(z)$ are given in equation (35) and (36). We shall now use the normalizing condition $P(1) = 1$ to determine the unknown $Q_{0,0}$ which appears in (37). Substituting $z = 1$ in (37) and using L'Hospitals rule, we obtain

$$Q_{0,0} = \frac{1-[p+\lambda E(X)E[S_b]+q(1-B^*(\lambda))E(X)]}{[1-p]\left[\frac{\lambda-\lambda qU(z_1)+\eta}{\eta} - \frac{[\lambda+\eta-qU(z_1)[\lambda+\eta A^*(\lambda+\eta)]]\left[\lambda E[S_b]S_v^*(\eta)+(1-B^*(\lambda))[1-pS_v^*(\eta)]\right]}{(\lambda+\eta)[1-pS_v^*(\eta)]-qS_v^*(\eta)[\lambda+\eta A^*(\lambda+\eta)]}\right]} \tag{38}$$

4. PARTICULAR CASES

Case (i):

If no customer receives the feedback service then by setting $p = 0$ in (37),

we get $P(z) = P_V(z) + P_B(z)$ (39)

$P_V(z) = \frac{N_V(z)}{D_V(z)} Q_{0,0}$ and $P_B(z) = \frac{N_B(z)}{D_B(z)} Q_{0,0}$

$$N_V(z) = \lambda[1 - A^*(\lambda + \eta)][\lambda - \lambda X(z) + \eta][X(z)S_v^*(\lambda - \lambda X(z) + \eta) - U(z_1)z] + \lambda[1 - S_v^*(\lambda - \lambda X(z) + \eta)][(\lambda + \eta)X(z) - U(z_1)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]] + [\lambda - \lambda X(z) + \eta][(\lambda + \eta)z - S_v^*(\lambda - \lambda X(z) + \eta)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]$$

$$D_V(z) = [\lambda - \lambda X(z) + \eta][(\lambda + \eta)z - S_v^*(\lambda - \lambda X(z) + \eta)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]$$

$$N_B(z) = \lambda\eta z[1 - S_v^*(\lambda - \lambda X(z) + \eta)][(\lambda + \eta)X(z) - U(z_1)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]][1 - S_b^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda)X(z)]] + \lambda\eta[1 - A^*(\lambda + \eta)][\lambda - \lambda X(z) + \eta][X(z)S_v^*(\lambda - \lambda X(z) + \eta) - U(z_1)z][z + [1 - z - S_v^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda)X(z)] - \lambda[1 - U(z_1)][\lambda - \lambda X(z) + \eta][(\lambda + \eta)z - S_v^*(\lambda - \lambda X(z))[\lambda X(z) + (\lambda - \lambda X(z) + \eta)]][z + [1 - z - S_b^*(\lambda - \lambda X(z))][B^*(\lambda) + (1 - B^*(\lambda)X(z)]]$$

$$D_B(z) = [\lambda - \lambda X(z)][\lambda - \lambda X(z) + \eta][z - S_b^*(\lambda - \lambda X(z))[B^*(\lambda) + (1 - B^*(\lambda)X(z)]][(\lambda + \eta)z - S_v^*(\lambda - \lambda X(z) + \eta)[\lambda X(z) + (\lambda - \lambda X(z) + \eta)A^*(\lambda + \eta)]]$$

and

$$Q_{0,0} = \frac{1-\lambda E(X)E[S_b]-(1-B^*(\lambda))E(X)}{\frac{\lambda-\lambda U(z_1)+\eta}{\eta} - \left[\frac{\lambda+\eta-U(z_1)[\lambda+\eta A^*(\lambda+\eta)]}{\lambda+\eta-S_v^*(\eta)[\lambda+\eta A^*(\lambda+\eta)}\lambda E(S_b)S_v^*(\eta)+(1-B^*(\lambda))\right]} \tag{40}$$

Equation (39) is well known generating function of the orbit size distribution of A bulk arrival retrial queue with exponentially distributed multiple working vacation studied by S.Pazhani Bala Murugan and R.Vijaykrishnaraj[15] irrespective of the notation.

Case (ii):
If there is no retrial then on setting $B^*(\lambda) = 1$, $A^*(\lambda + \eta) = 1$ in (37), we get

$$P(z) = P_V(z) + P_B(z) \tag{41}$$

$$P_V(z) = \frac{N_V(z)}{D_V(z)} Q_{0,0} \text{ and } P_B(z) = \frac{N_B(z)}{D_B(z)} Q_{0,0}$$

$$N_V(z) = \lambda[1 - S_v^*(\lambda - \lambda X(Z) + \eta)][X(z) - qU(z_1)z] + [\lambda - \lambda X(z) + \eta][z - (q + pz)S_v^*(\lambda - \lambda X(z) + \eta)]$$

$$D_V(z) = [\lambda - \lambda X(z) + \eta][z - (q + pz)S_v^*(\lambda - \lambda X(z) + \eta)]$$

$$N_B(z) = \lambda\eta z[1 - S_v^*(\lambda - \lambda X(z) + \eta)][X(z) - qU(z_1)z][1 - S_b^*(\lambda - \lambda X(z))] - \lambda[1 - qU(z_1)][\lambda - \lambda X(z) + \eta][z - (q + pz)S_v^*(\lambda - \lambda X(z) + \eta)][1 - S_b^*(\lambda - \lambda X(z))]$$

$$D_B(z) = [\lambda - \lambda X(z)][\lambda - \lambda X(z) + \eta][z - (q + pz)S_b^*(\lambda - \lambda X(z))][z - (q + pz)S_b^*(\lambda - \lambda X(z) + \eta)]$$

$$Q_{0,0} = \frac{1 - [p + \lambda E(X)E(S_b)]}{[1-p]\left[\frac{\lambda - \lambda qU(z_1) + \eta}{\eta} - \frac{1 - qU(z_1)}{1 - S_v^*(\eta)}[\lambda E(S_b)S_v^*(\eta)]\right]} \tag{42}$$

Equation (41) is well known generating function of the queue size distribution of an $M^X/G/1$ queue with feedback and multiple working vacation studied by K.Santhi and S.Pazhani Bala Murugan[14] irrespective of the notation.

Case(iii):
If no customer receives the feedback service and no retrial then on setting $A^*(\lambda + \eta) = 1$, $B^*(\lambda)$ and $p = 0$ in (37), we get

$$P(z) = P_V(z) + P_B(z) \tag{43}$$

$$P_V(z) = \frac{N_V(z)}{D_V(z)} Q_{0,0} \text{ and } P_B(z) = \frac{N_B(z)}{D_B(z)} Q_{0,0}$$

$$N_V(z) = \lambda[1 - S_v^*(\lambda - \lambda X(z) + \eta)][X(z) - U(z_1)z] + [\lambda - \lambda X(z) + \eta][z - S_v^*(\lambda - \lambda X(z) + \eta)]$$

$$D_V(z) = [\lambda - \lambda X(z) + \eta][z - S_v^*(\lambda - \lambda X(z) + \eta)]$$

$$N_B(z) = \lambda\eta z[1 - S_v^*(\lambda - \lambda X(z) + \eta)][X(z) - U(z_1)z][1 - S_b^*(\lambda - \lambda X(z))]$$

$$D_B(z) = [\lambda - \lambda X(z)][z - S_b^*(\lambda - \lambda X(z)][\lambda - \lambda X(z) + \eta][z - S_v^*(\lambda - \lambda X(z) + \eta)]$$

$$Q_{0,0} = \frac{1 - \lambda E(X)E(S_b)}{\left[\frac{\lambda - \lambda U(z_1) + \eta}{\eta} - \frac{1 - U(z_1)}{1 - S_v^*(\eta)}[\lambda E(S_b)S_v^*(\eta)]\right]} \tag{44}$$

Equation (43) is well known generating function of the queue size distribution of an $M^X/G/1$ queue with multiple working vacation studied by M.I.Aftab Begum (2011) irrespective of the notation.

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