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# Equitable Total Coloring of Direct Product of Path and Cycle

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**ABSTRACT**

An equitable total coloring of graph $G$ is an assignment of colors to all the elements (vertices, edges) of graph $G$ such that adjacent or incident elements receive the different color and for any two color classes different by at most one. In this paper, we obtain the exact expressions for the equitable total coloring of direct product of path and cycle.

**Mathematics Subject Classification:** 05C15.

**Keywords:** Equitable total coloring, Tensor product, Direct product and cycle graph.

## 1. INTRODUCTION

All graphs considered finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$ respectively. A total coloring of graph $G$ is a coloring of the vertices and the edges of $G$ such that any two adjacent or incident elements (vertices, edges) have different color. The total chromatic number of a graph $G$ denoted by $\chi''(G)$, is the minimum number of colors that required in a total coloring. Total Coloring Conjecture formulated by Behzad[1] and Vizing[9], says that $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$ for a simple graph $G$. A total coloring of graph $G$ is said to be equitable if the number of elements (vertices, edges) are colored with each color differ by at most one. The minimum number of colors that required for an equitable total coloring of graph $G$ is called the equitable total chromatic number of $G$ and is denoted by $\chi''_e(G)$. In 1973, Meyer[6] introduced the concepts of an equitable coloring and the equitable chromatic number. An equitable total coloring of $G$ is a mapping $f: V(G) \cup E(G) \rightarrow C$, where $C$ is a set of colors satisfying the following conditions.

1. $f(u) \neq f(v)$ for any two adjacent vertices $u, v \in V(G)$,
2. $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$,
3. $f(v) \neq f(e)$ for any vertex $v \in V(G)$ and any edge $e \in E(G)$ incident to $v$ and
4. $||T_i| - |T_j|| \leq 1;\ i, j = 1, 2, \ldots, k.\ T_i = V_i \cup E_i;\ 1 \leq i \leq k.$

In 1994, Fu[5] investigated the concept of an equitable total coloring and the equitable total chromatic number. He raised the conjecture that for any simple graph G, then $\chi_e''(G) \leq \Delta(G) + 2$. Graph products were first defined by Sabidussi[8] and Vizing[10]. A lot of work was done on various topics related to graph product, but on the other hand there are many open questions. Pranaver and Zmazek[7] proved that total chromatic number of $(P_m \times P_n)$ and $(C_m \times P_n)$ are 5. Geetha and Somasundaram[3] proved the TCC for direct product of some graphs.

In this paper, we found the expressions for the equitable total chromatic number of direct product of path and cycle.

## 2. RESULTS AND DISCUSSION

**Definition 2.1:**[4] The direct product of $G$ and $H$ is a graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$ and for which vertices $(g, h)$ and $(g', h')$ are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. In other words, $V(G \times H) = \{(g, h) |\ g \in V(G)\ and\ h \in V(H)\}$ and $E(G \times H) = \{((g, h), (g', h')) | gg' \in E(G) and\ hh' \in E(H)\}$.

**Theorem 2.1:** [2] For any cycle graph $C_n$,

$$\chi_e''(C_n) = \begin{cases} \Delta(G) + 1, & if\ n \equiv 0\ mod\ 3 \\ \Delta(G) + 2, & otherwise. \end{cases}$$

**Theorem 2.2:** The equitable total chromatic number of $P_n \times P_m$, for all $n, m \geq 3$ and n, m $\in Z^+$

$$\chi_e''(P_n \times P_m) = \begin{cases} \Delta(P_n \times P_m) + 1, & n = 4 + 5k, m \geq 3, k = 0,1,2 \ldots. \\ \Delta(P_n \times P_m) + 2, & for\ all\ n, m \geq 3 \end{cases}$$

**Proof:** Let $P_n$ be path on $n$ vertices $\{u_1, u_2, u_3, \ldots, u_n\}$ and $P_m$ be path on $m$ vertices $\{v_1, v_2, v_3, \ldots, v_m\}$ respectively. Then the direct product of $(P_n \times P_m)$ divide into two cases as follows:

**Case (1) : If $n = 4 + 5k,\ m \geq 3$**

Here, $\Delta(P_n \times P_m) + 1$ is the equitable total chromatic number of $(P_n \times P_m)$. In this graph of direct product having $n(R_1, R_2, \ldots, R_n)$ rows and $m\ (C_1, C_2, \ldots, C_m)$ columns.

Assign the colors to all the vertices of $R_1, R_6, R_{11}, \ldots, R_{n-3}$ with the color 1 and all the vertices of $R_2, R_7, R_{12}, \ldots, R_{n-2}$ with the color 2 and all the vertices of $R_3, R_8, R_{13} \ldots, R_{n-1}$ with the color 3 and all the vertices of $R_4, R_9, R_{14}, \ldots, R_n$ with the color 4 and all the vertices of $R_5, R_{10}, R_{15}, \ldots, R_{n-4}$ with the color 5. To color the edges between $(R_2, R_3)$ assign the colors of $R_1 and\ R_4$. Then to color the edges between $(R_3, R_4)$ assign the colors of $R_2\ and\ R_5$. Use the above process to color all the edges between $((R_4, R_5), (R_5, R_6), \ldots, (R_{n-2}, R_{n-1}))$.

Finally, we color the remaining uncolored edges between $(R_1, R_2)$ $and$ $(R_{n-1}, R_n)$ using the missing colors which satisfies the condition of equitably total colorable.
Hence, $\chi''_e(P_n \times P_m) = \Delta\,(P_n \times P_m) + 1$.

**Case (2) : $\boldsymbol{For\ all\ n, m \geq 3}$**
Here, $\Delta\,(P_n \times P_m) + 2$ is the equitable total chromatic number of $(P_n \times P_m)$. In this graph of direct product having $n(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta\,(P_n \times P_m) + 2$ into three equal color classes, say $X_1, X_2$ $and$ $X_3$ respectively.

**Subcase (2.1) : If $\boldsymbol{n \equiv 0 (mod\ 3)}$**
Color all the edges between $R_1$ $and$ $R_2$ using $X_1$ color and the edges between $R_2$ $and$ $R_3$ using $X_2$ color and the edges between $R_3$ $and$ $R_4$ using $X_3$ color. Then we color the remaining edges between $((R_4, R_5), (R_5, R_6), \dots, (R_{n-1}, R_n))$ of $(P_n \times P_m)$ using the colors of $X_1$, $X_2$ $and$ $X_3$ cyclically. Now at each vertex in a row, we have exactly one missing color classes. Using the missing color class, we give color to all the vertices in $R_2, R_3, \dots, R_{n-1}$. Finally, we color the remaining vertices of $R_1$ using $X_2$ $and$ $X_3$ colors and vertices of $R_n$ using $X_1$ $and$ $X_3$ colors according to satisfying the equitably total colorable conditions.

**Subcase (2.2) : If $\boldsymbol{n \equiv 1 (mod\ 3)}$**
By subcase 2.1, assign colors to all the edges between $((R_1, R_2), (R_2, R_3), \dots, (R_{n-1}, R_n))$ of $(P_n \times P_m)$. Now, we color all the vertices of $R_2, R_3, \dots, R_{n-1}$ using the colors of $X_3$, $X_1$ $and$ $X_2$ cyclically. Finally, we color the remaining uncolored vertices of $R_1$ using $X_2$ $and$ $X_3$ colors and vertices of $R_n$ using $X_1$ $and$ $X_2$ colors according to satisfying the equitably total colorable conditions.

**Subcase (2.3) : If $\boldsymbol{n \equiv 2 (mod\ 3)}$**
Assign $X_1$ color to all the edges between $R_1$ $and$ $R_2$ and $X_2$ color to all the edges between $R_2$ $and$ $R_3$ and $X_3$ color to all the edges between $R_3$ $and$ $R_4$. Then we color the remaining edges between $((R_4, R_5), (R_5, R_6), \dots, (R_{n-2}, R_{n-1}))$ of $(P_n \times P_m)$ using the colors of $X_1$, $X_2$ $and$ $X_3$ cyclically. Now at each vertex in a row, we have exactly one missing color classes. Using the missing color class, we give color to all the vertices of $R_2, R_3, \dots, R_{n-2}$. Then to color all the vertices of $R_{n-1}$ assign the colors of $R_{n-2}$ and assign $X_2$ color to all the edges between $(R_{n-1}, R_n)$. Finally, we color the remaining vertices of $R_1$ using $X_2$ $and$ $X_3$ colors and vertices of $R_n$ using $X_1$ $and$ $X_3$ colors according to satisfying the equitably total colorable conditions.
Therefore, The equitable total chromatic number of $(P_n \times P_m)$ is $\Delta\,(P_n \times P_m) + 2$.

**Theorem 2.3:** The equitable total chromatic number of $C_n \times C_m$, for all $m \geq 3$ and n, m ∈ $Z^+$

$$\chi''_e(C_n \times C_m) = \begin{cases} \Delta\,(C_n \times C_m) + 1, & n = 5k, \quad k = 1, 2, 3, \dots \\ \Delta\,(C_n \times C_m) + 2, & n = 2l, 3k, \ l = 2, 3, 4, \dots \end{cases}$$

**Proof:** Let $C_n$ be the cycle on $n$ vertices $\{u_1, u_2, u_3, \dots, u_n\}$ and $C_m$ be the cycle on $m$ vertices $\{v_1, v_2, v_3, \dots, v_m\}$ respectively. Then the direct product of $(C_n \times C_m)$ divide into two cases as follows:

**Case (1) : If $n = 5k$ and $m \geq 3$**

Here, $\Delta\,(C_n \times C_m) + 1$ is the equitable total chromatic number of $(C_n \times C_m)$. In this graph of direct product having $n(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns.

Assign the colors to all the vertices of $R_1, R_6, R_{11}, \dots, R_{n-4}$ with the color 1 and all the vertices of $R_2, R_7, R_{12}, \dots, R_{n-3}$ with the color 2 and all the vertices of $R_3, R_8, R_{13}, \dots, R_{n-2}$ with the color 3 and all the vertices of $R_4, R_9, R_{14}, \dots, R_{n-1}$ with the color 4 and all the vertices of $R_5, R_{10}, R_{15} \dots, R_n$ with the color 5. To color the edges between $(R_1, R_2)$ assign the colors of $R_n$ *and* $R_3$. Then to color the edges between $(R_2, R_3)$ assign the colors of $R_1$ *and* $R_4$. Use the above process to color all the edges between $((R_3, R_4),\ (R_4, R_5), \dots, (R_n, R_1))$ which satisfies the condition of equitably total colorable.

Hence, $\chi''_e(C_n \times C_m) = \Delta\,(C_n \times C_m) + 1$.

**Case (2):**

Here, $\Delta\,(C_n \times C_m) + 2$ is the equitable total chromatic number of $(C_n \times C_m)$. In this graph of direct product having $n(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta\,(C_n \times C_m) + 2$ into three equal color classes, say $X_1, X_2$ *and* $X_3$ respectively.

**Subcase (2.1) : If $n = 2l$ and $m \geq 3$**

Color all the vertices of $R_1, R_3, R_5 \dots, R_{n-1}$ using one of the color of $X_1$ and assign remaining color of $X_1$ to all the vertices of $R_2, R_4, R_6 \dots, R_n$. Now, color all the edges between $R_1$ *and* $R_2$ using $X_2$ color and the edges between $R_2$ *and* $R_3$ using $X_3$ color. Then we color all the edges between $((R_3, R_4),\ (R_4, R_5), \dots, (R_n, R_1))$ using $X_2$ *and* $X_3$ cyclically which satisfies the condition of equitably total colorable.

**Subcase (2.2) : If $n = 3k$ and $m \geq 3$**

Color all the edges between $R_1$ *and* $R_2$ using $X_1$ color and the edges between $R_2$ *and* $R_3$ using $X_2$ color and the edges between $R_3$ *and* $R_4$ using $X_3$ color. Then we color the remaining edges between $((R_4, R_5),\ (R_5, R_6), \dots, (R_n, R_1))$ of $(C_n \times C_m)$ using the colors of $X_1$, $X_2$ *and* $X_3$ cyclically. Now, we color all the vertices of $R_1, R_2, \dots, R_n$ using the colors of $X_2$, $X_3$ *and* $X_1$ cyclically which satisfies the condition of equitably total colorable.

Therefore, The equitable total chromatic number of $(C_n \times C_m)$ is $\Delta\,(C_n \times C_m) + 2$.

**Theorem 2.4:** The equitable total coloring of $P_n \times C_m$, for all $n, m \geq 3$ and n, m $\in Z^+$

$$\chi''_e(P_n \times C_m) = \begin{cases} \Delta\,(P_n \times C_m) + 1, & n = 4 + 5k, m \geq 3, k = 0,1,2 \dots. \\ \Delta\,(P_n \times C_m) + 2, & for\ all\ n, m \geq 3 \end{cases}$$

**Proof:** Let $P_n$ be the path on $n$ vertices $\{u_1, u_2, u_3, \dots, u_n\}$ and $C_m$ be the cycle on $m$ vertices $\{v_1, v_2, v_3, \dots, v_m\}$ respectively. Then the direct product of $(P_n \times C_m)$ divide into two cases as follows:

**Case (1): If $n = 4 + 5k$ and $m \geq 3$**

Here, $\Delta\,(P_n \times C_m) + 1$ is the equitable total chromatic number of $(P_n \times C_m)$. In this graph of direct product having $n(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns.

Colors all the vertices of $R_1, R_6, R_{11}, \dots, R_{n-3}$ with the color 1 and all the vertices of $R_2, R_7, R_{12}, \dots, R_{n-2}$ with the color 2 and all the vertices of $R_3, R_8, R_{13} \dots, R_{n-1}$ with the color

3 and all the vertices of $R_4, R_9, R_{14}, \dots, R_n$ with the color 4 and all the vertices of $R_5, R_{10}, R_{15}, \dots, R_{n-4}$ with the color 5. To color the edges between $(R_2, R_3)$ assign the colors of $R_1 and\ R_4$. Then to color the edges between $(R_3, R_4)$ assign the colors of $R_2 and\ R_5$. Use the above process to color all edges between $((R_4, R_5),\ (R_5, R_6), \dots, (R_{n-2}, R_{n-1}))$. Finally, we color the uncolored edges between $(R_1, R_2)\ and\ (R_{n-1}, R_n)$ using the remaining colors which satisfies the condition of equitably total colorable.

Hence, $\chi''_e(P_n \times C_m) = \Delta\,(P_n \times C_m) + 1$.

**Case (2):** $\boldsymbol{For\ all\ n, m \geq 3}$

Here, $\Delta\,(P_n \times C_m) + 2$ is the equitable total chromatic number of $(P_n \times C_m)$. In this graph of direct product having $n(R_1, R_2, \dots, R_n)$ rows and $m\,(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta\,(P_n \times C_m) + 2$ into three equal color classes, say $X_1, X_2\ and\ X_3$ respectively.

**Subcase (2.1): If** $\boldsymbol{n \equiv 0 (mod\ 3)\ and\ m = 3}$

Color all the vertices of $R_2$ using one of the color of $X_3$ and assign remaining color of $X_3$ to all the vertices of $R_{n-1}$. Then color all the vertices of $R_3, R_4, \dots, R_{n-2}$ using the colors of $X_1$, $X_2\ and\ X_3$ cyclically. Now, color all the edges between $R_1\ and\ R_2$ using $X_1$ color and the edges between $R_2\ and\ R_3$ using $X_2$ color and the edges between $R_3\ and\ R_4$ using $X_3$ color. Then the remaining edges between $((R_4, R_5),\ (R_5, R_6), \dots, (R_{n-1}, R_n))$ of $(P_n \times C_m)$ using the colors of $X_1,\ X_2\ and\ X_3$ cyclically. Finally, color the remaining vertices in $R_1$ and $R_n$ using $\Delta\,(P_n \times C_m) + 1$ colors according to satisfying the equitably total colorable condition.

**Subcase (2.2): If** $\boldsymbol{n \equiv 0 (mod\ 3)\ and\ m > 3}$

Color all the edges between $R_1\ and\ R_2$ using $X_1$ color and the edges between $R_2\ and\ R_3$ using $X_2$ color and the edges between $R_3\ and\ R_4$ using $X_3$ color. Then we color all the remaining edges between $((R_4, R_5),\ (R_5, R_6), \dots, (R_{n-1}, R_n))$ of $(P_n \times C_m)$ using the colors of $X_1$, $X_2\ and\ X_3$ cyclically. Now at each vertex in a row, we have exactly one missing color classes. Using the missing color class, we give color to all the vertices in $R_2, R_3, \dots, R_{n-1}$. Finally, we color the remaining uncolored vertices of $R_1$ using $X_2\ and\ X_3$ colors and vertices of $R_n$ using $X_1\ and\ X_3$ colors according to satisfying the equitably total colorable condition.

**Subcase (2.3): If** $\boldsymbol{n \equiv 1 (mod\ 3)\ and\ m \geq 3}$

By subcase 2.2, assign colors to all the edges between $((R_1, R_2),\ (R_2, R_3), \dots, (R_{n-1}, R_n))$ of $(P_n \times C_m)$. Now, we color all the vertices of $R_2, R_3, \dots, R_{n-1}$ using the colors of $X_3$, $X_1\ and\ X_2$ cyclically. Finally, we color the remaining uncolored vertices of $R_1$ using $X_2\ and\ X_3$ colors and vertices of $R_n$ using $X_1\ and\ X_2$ colors according to satisfying the equitably total colorable condition.

**Subcase (2.4): If** $\boldsymbol{n \equiv 2 (mod\ 3)\ amd\ m = even}$

Assign $X_1$ color to all the edges between $R_1\ and\ R_2$ and $X_2$ color to all the edges between $R_2\ and\ R_3$ and $X_3$ color to all the edges between $R_3\ and\ R_4$. Then we color all the remaining edges between $((R_4, R_5),\ (R_5, R_6), \dots, (R_{n-2}, R_{n-1}))$ of $(P_n \times C_m)$ using the colors of $X_1$, $X_2\ and\ X_3$ cyclically. Now, we color all the vertices of $R_2, R_3, \dots, R_{n-2}$ using the colors of $X_3,\ X_1\ and\ X_2$ cyclically. Then assign $X_1$ colors to all the vertices of $R_{n-1}$ and assign $X_2$

color to all the edges between $(R_{n-1}, R_n)$. Finally, we color the remaining vertices of $R_1$ using $X_2$ *and* $X_3$ colors and vertices of $R_n$ using $X_1$ *and* $X_3$ colors according to satisfying the equitably total colorable condition.

**Subcase (2.5): If $n \equiv 2(mod\ 3)$ $amd$ $m = odd$**

Color all the edges between $R_1$ *and* $R_2$ using $X_1$ color and the edges between $R_2$ *and* $R_3$ using $X_2$ color and the edges between $R_3$ *and* $R_4$ using $X_3$ color. Then we color the remaining edges between $((R_4, R_5), (R_5, R_6), \ldots, (R_{n-3}, R_{n-2}))$ of $(P_n \times C_m)$ using the colors of $X_1$, $X_2$ *and* $X_3$ cyclically. Now, we color all the vertices of $R_2, R_3, \ldots, R_{n-3}$ using the colors of $X_3$, $X_1$ *and* $X_2$ cyclically. Then, color the vertex of $R_{n-2}$ using one of the color of $X_1$ from $C_1$to $C_{\frac{m-1}{2}}$ and use remaining color of $X_1$ from $C_{\frac{m+3}{2}}$ to $C_m$ and color $C_{\frac{m+1}{2}}$ using one of the color of $X_3$. Then, color the vertex of $R_{n-1}$ using one of the color of $X_1$ from $C_2$to $C_{\frac{m-1}{2}}$ and use remaining color of $X_1$ from $C_{\frac{m+3}{2}}$ to $C_{m-1}$ and color $C_1, C_m$ *and* $C_{\frac{m+1}{2}}$ using $X_2$ colors suitably. Now, we color the edges between $(R_{n-2}, R_{n-1})$ using $X_1$and $X_3$ colors and the edges between $(R_{n-1}, R_n)$ using $X_1$and $X_2$ colors. Finally, we color the remaining vertices of $R_1$ using $X_2$ *and* $X_3$ colors and vertices of $R_n$ using $X_1$ *and* $X_3$ colors according to satisfying the equitably total colorable condition.

Therefore, The equitable total chromatic number of $(P_n \times C_m)$ is $\Delta\,(P_n \times C_m) + 2$.

## CONCLUSION

In this paper, we have obtained the tight bound an equitable total chromatic number of the direct product of path by path and path by cycle. Further, the extension of this result for strong product of path, cycle and complete graph.

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