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Generalized Fixed Point Theory of Cone B-matric Spaces

S. K. Tiwari[1] and Rukhamani Kurre[2]

[1]Prof. & Head
[2]Department of Mathematics
Dr.C.V.Raman University,kota,Bilaspur(C.G)

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ABSTRACT

In this paper, In this paper, we establish the relationship amongst metric space, b-metric space, cone metric space and cone b-metric space. We show that the cone b-metric space generalizes the metric space, b-metric space and cone metric space.

Keywords: b-metric, cone metric space, cone b-metric space, fixed point theorem.

1.1 INTRODUCTION

A generalization of metric spaces by introducing the concept of cone metric space. They used an ordered Banach space in place of set of real numbers in metric space. They also obtained some fixed point theorems in this space for mappings satisfying various types of contractive conditions. 1992 saw the work of S. Banach known as the Banach Contraction Principle, which laid the foundation for studies of Fixed Point Theory. Since many mathematician have extended and generalised the work of Banach in different directions. In 1976 Taskovic proved a location theorem on cartesian product of metric spaces as a solution of kartowski's problem of 1932. Later many such results were proved many author including Presic, Ciric and Presic *et al.*,[1] introduced the concept of b-metric space as a generalisation of a metric space and proved the contraction mapping theorem in a b-metric space . Since then several papers have dealt with fixed point theory or variational principle for single valued and multivalued mapping in b-metric . Huang and Zang[2] generalizing the notation of a metric space by replacing the set of real numbers by ordered spaces, defined a cone metric space and proved some fixed point theorems of contractive mapping defined on these spaces. Rezapour and Hamlbarani[3] omitting the assumption of normality obtained by generalization of results. In 2008 Di Bari and Verto[4] obtained results on points of coincidence and common fixed in non-normal cone metric spaces. Further results on fixed point theorems in such spaces were obtain by several authors. In Hussein and Shah introduced cone b-metric spaces as a generalisation b-metric spaces and cone metric spaces, established some topological properties in such spaces and improved some results of KKM mapping in the setting of a cone b-metric space. Cone b-metric spaces play useful role in fixed point theory. In fact there exist mapping with common fixed points which are contraction mapping in a cone b-metric space but are not contraction mapping when defined in a cone metric space.

1.2 B-METRIC SPACE

Let X be a set and R^+ denote the set of all non-negative numbers. A function is said to an b-metric iff the following conditions are satisfied:

(i) d(x, y) = 0 iff x = y,
(ii) d(x, y) = d(y, x),
(iii) d(x, y) < r and d(x, z) < r imply d(y, z) < 2r.

Then the pair (X, d) is called an b-metric space.

Note: The usual metric space is a b-metric space but the converse needs not be true.

Example 1. Let X = 0, 1, 2 and d(2, 0) = d(0, 2) = m ≥ 2, d(0, 1) = d(1, 2) = d(0, 1) = d(2, 1) = 1 and d(0, 0) = d(1, 1) = d(2, 2) = 0. Then
d(x, y) ≤ $\frac{m}{2}$[d(x, z) +d(y, z)] for all x, y, z ∈ X. If m > 2, the ordinary triangle inequality does not hold.

The definitions of convergence, of Cauchy sequence in b-cone are as follows:

Def 1. Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in
X is called a Cauchy sequence if and only if for $\epsilon > 0$, there exists n$c \epsilon$ N
such that for each m, n ≥ n(ϵ), we have d(xn, xm) < ϵ.

Def 2. Let (X, d) be a b-metric space. Then a sequence {xn} in X is called a convergent sequence if and only if there exists x ∈ X, such that there exists n(ϵ) ∈ N such that for all n ≥ n(ϵ) we have d(xn, x) < ϵ. We denote this by lim xn $\lim_{n\to\infty}$ xn $= x$.

Def 3. The b-metric space is complete if every Cauchy sequence in it is convergent.

1.3 CONE METRIC SPACE

Let X be a nonempty set. Suppose the mapping d : X × X → E satisfies the following conditions:

(i) 0 < d(x, y) for all x, y ∈ X and d(x, y) = 0 iff x = y,
(ii) d(x, y) = d(y, x) for all x, y ∈ X,
(iii) d(x, y) ≤ d(x, z) + d(z, y) for all x, y, z ∈ X.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 2. Let E = R2, P = {(x, y) : x, y ∈ X, x, y ≥ 0} ∈R2,X = R and d : X × X → E such that d(x, y) = (|x − y|, α |x − y|), where α≥ 0 is a constant. Then (X, d) is a cone metric space.
The definitions of convergence, Cauchy sequence in b-cone are as follows:

Def 4. Let (X, d) be a cone metric space. Let {xn} be a sequence in X and x ∈ X. Then {xn} converges to x if for every c ∈ M with 0 ∈ c, there is an integer Nc such that for all n ≥ Nc, d(xn, x) ∈ c. We denote this by limn!1 xn = x, or xn → ∞ as n → ∞.

Def 5. Let (X, d) be a cone metric space. Let {xn} be a sequence in X and x ∈ X. Then {xn} converges to x if for every c ∈ M with 0 ∈ c, there is an integer N such that for all n,m ≥ N, d(xn, xm) ∈c. Then {xn} is called a Cauchy sequence in X.

Def 6. Let (X, d) be a cone metric space. Then X is called a complete cone metric space if every Cauchy sequence in X is convergent in X.

Def 7. Let (X, d) be a cone metric space. If for any sequence {xn}in X, there is a subsequence {xni} of X such that {xni} is convergent in X.
Then X is called sequentially compact cone metric space.
Remark 2. The cone metric space generalizes the metric space. But the converse needs not to be true.

1.4. CONE B-METRIC SPACE

Let X be nonempty set and E a real Banach space with cone P. A vector valued function D : X × X → P is said to be a cone b-metric function on X with constant K ≥ 1, if the following conditions are satisfied:
(i) _ ≤ D(x, y) for all x, y ∈ X and D(x, y) = _ if and only if x = y,
(ii) D(x, y) = D(y, x) for all x, y ∈ X,
(iii) D(x, z) ≤ K(D(x, y) + D(y, z)) for all x, y, z ∈X.
The pair (X,D) is called the cone b-metric space. By _ we denote the zero element of E.
Note: Every cone metric space is a cone b-metric space, but the Converse need not to be true.

Example 3. Let E = R2, P = {(x, y) : x, y ∈ X, x, y ≥ 0} αR2,X = R
and d : X × X → E such that d(x, y) = (|x − y|p, α |x − y|p) , where _ ≥ 0
and p > 1 are two constants. Then it is a cone b-metric space, but not a cone metric space.
Note: Every b-metric space is a b-cone metric space. But the converse needs not to be true.

The definitions of convergence and Cauchy sequence in cone are as follows:

Def 8. Let (X,D) be b-cone metric space. We say that {xn} is:
(i) a Cauchy sequence, if for every c∈E with α ∈ c there is N such that for all m, n > N,D(xn, xm) ∈ c,
(ii) convergent sequence, if for every c ∈ E with α ∈ c there is N such that for all n > N,D(xn, x) ∈ c for some fixed x ∈X. A cone b-metric space X is said to be complete, if every Cauchy sequence in it is convergent in X.
(iii) (X,D) is a complete cone b-metric space, if every Cauchy sequence is convergent.

Proposition 1. Let (X,D) be b-cone metric space, P be a normal cone with normal constant K, x ∈ X and {xn} a sequence in X. Then:
(i) {xn} converges if and only if D(xn, x) → 0,
(ii) the limit point of every sequence is unique,
(iii) every convergent sequence is Cauchy,
(iv) {xn} is a Cauchy sequence if D(xn, xm) → 0 as m, n → ∞.

1.4 MAIN RESULTS

Theorem 1. Let (X, d) be a complete cone b-metric space with coefficient s ≥ 1 and solid cone P contained in a real Banach space E. For any positive integer k, let S, $T : X^k \to X$ and be mappings satisfying the following conditions:

$$S(X^k) \bigcup T(X^k) \subseteq f(X) \tag{1.4.1}$$

$$\max_{\substack{\beta=SVT \\ \gamma=SVT}} \{d(\beta(x_1, x_2, \ldots\ldots, x_k), \gamma(x_2, x_3, \ldots\ldots\ldots, x_{k+1})\} \leq \lambda\phi(d(fx_1, fx_2), d(fx_2, fx_3), \ldots\ldots, d(fx_k, x_{k+1})) \tag{1.4.2}$$

for all $x_1, x_2, \ldots\ldots, x_{k+1} \in X, S^k \lambda \in (0,1)$

f(x) is completed.

there exist element $x_1, x_2, \ldots\ldots, x_k, x_{k+1} in\, X$ and R in E such that $fx_{k+1} = T(x_1, x_2, \ldots\ldots x_k)$, 0<<R and R is upper bound of the set

$$\left\{ \frac{d(fx_1, fx_2)}{\theta}, \frac{d(fx_2, fx_3)}{\theta^2}, \ldots\ldots\ldots \frac{d(fx_k, fx_{k+1})}{\theta^k} \right\} \tag{1.4.3}$$

Where $\theta = \lambda^{\frac{1}{k}}$

Then f and T have a coincidence point,
i.e.,

$$C(f, T) \neq 0$$

Proof: we define a sequence $< y_n >$ in $f(X)$ by $y_n = fx_n$ for
$n = 1,2,\ldots.., k,\ y_{k+2n-1} = f(x_{k+2n-1}) = S(x_{2n-1}, x_{2n}, \ldots\ldots\ldots, x_{2n+k-2})$ and
$y_{k+2n} = f(x_{k+2n}) = T(x_{2n}, x_{2n+1}, \ldots\ldots\ldots, x_{2n+k=1})\, for\ n = 1,2,\ldots\ldots$

Let $\alpha_n = d(y_n, y_{n+1})$. Then by the mathematical induction, we will now prove that

$$\alpha_n \leq R.\theta^n \tag{1.4.5}$$

For all n. clearly by the definition of R is true for n=1,2,….,*k*. let the k inequalities $\alpha_n \leq R.\theta^n$, $\alpha_{n+1} \leq R\theta^{n+1}, \ldots\ldots\ldots, \alpha_{n+k-1} \leq R\theta^{n+k-1}$ be the include hypothesis. Then for only odd n we have

$$\begin{aligned}\alpha_{k+n} &= d(y_{k+n}, y_{k+n+1}) \\ &= d(S(x_n, x_{x+1}, \ldots\ldots x_{n+k-1}), T(x_{n+1}, x_{n+2}, \ldots\ldots\ldots, x_{n+k})) \\ &\le \lambda\phi(d(fx_n, fx_{n+1}), d(fx_{n+1}, fx_{n+2}), \ldots\ldots\ldots, d(fx_{n+k-1}, fx_{n+k}) \\ &= \lambda\phi(\phi_n, \phi_{n+1}, \ldots\ldots\ldots\ldots\ldots\ldots, \phi_{n+k-1}) \\ &\le \lambda\phi(R\theta^n, R\theta^{n+1}, \ldots\ldots\ldots\ldots\ldots, R\theta^{n+k-1}) \\ &\le \lambda\phi(R\theta^n, R\theta^n, \ldots\ldots\ldots\ldots\ldots\ldots, R\theta^n) \le \lambda R\theta^n = R\theta^{n+k}\end{aligned}$$

Further

$$\begin{aligned}\alpha_{k+n+1} &= d(y_{k+n+1}, y_{k+n+2}) \\ &= d(S(x_{n+1}, x_{x+2}, \ldots\ldots x_{n+k}), T(x_{n+2}, x_{n+3}, \ldots\ldots\ldots, x_{n+k+1})) \\ &\le \lambda\phi(d(fx_{n+1}, fx_{n+2}), d(fx_{n+2}, fx_{n+3}), \ldots\ldots\ldots, d(fx_{n+k}, fx_{n+k+1}) \\ &= \lambda\phi(\phi_{n+1}, \phi_{n+2}, \ldots\ldots\ldots\ldots\ldots\ldots, \phi_{n+k}) \\ &\le \lambda\phi(R\theta^{n+1}, R\theta^{n+2}, \ldots\ldots\ldots\ldots\ldots, R\theta^{n+k}) \\ &\le \lambda\phi(R\theta^{n+1}, R\theta^{n+1}, \ldots\ldots\ldots\ldots\ldots\ldots, R\theta^{n+1}) \le \lambda R\theta^{n+1} = R\theta^{n+k+1}\end{aligned}$$

Thus inductive proof of [1.4.5] is completed.

Now for all $n, p \in N$, we have

$$\begin{aligned}d(y_n, y_{n+p}) &\le s.d(y_n, y_{n+1}) + s^2.d(y_{n+1}, y_{n+2}) + \ldots\ldots\ldots + s^p.d(y_{n+p-1}, y_{n+p}) \\ &\le sR\theta^n + s^2 R\theta^{n+1} + \ldots\ldots\ldots\ldots\ldots + s^p R\theta^{n+p-1} \\ &\le s^p(R\theta^n + R\theta^{n+1} + \ldots\ldots\ldots\ldots\ldots + R\theta^{n+p-1}) \\ &\le s^p R\theta^n(1+\theta+\theta^2 \ldots\ldots\ldots\ldots\ldots) \\ &= \frac{s^p R\theta^n}{1-\theta}\end{aligned}$$

Now let 0<<c be given. Choose $\delta > 0$ such that $c + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{y \in E; \|y\| < \delta$ .Also choose a natural number N_1 such that $\frac{s^p R\theta^n}{1-\theta} \in N_\delta(0)$ for all $n \ge N_1$. Then $\frac{s^p R\theta^n}{1-\theta} << c$ for all $n \ge N_1$ and so $d(y_n, y_{n+p}) \le \frac{s^p R\theta^n}{1-\theta}$ for all $n \ge N_1$.

Hence , the sequence $<y_n>$ is a Cauchy sequence in *f*(*X*) and since *f*(X) is complete, there exists $v, u \in X$ such that $\lim_{n\to\infty} y_n = v = f(u)$.

Now choose a natural number N_2 such that $d(y_n, y_{n+1}) << \dfrac{c}{s^{k+1}\lambda(k+1)}$ and

$d(v, y_n) << \dfrac{c}{s(k+1)}$ for all $n \geq N_2$. Suppose k is even. Then for all $n \geq N_2$

$$d(fu, S(u,u,.......)) \leq s[d(fu, y_{k+2n-1}) + d(y_{k+2n-1}, S(u,u,........))]$$
$$= s[d(fu, y_{k+2n-1}) + d(S(x_{2n-1}, x_{2n}, ..., x_{k+2n-2}), S(u,u,........))]$$
$$\leq sd(fu, y_{k+2n-1}) + s^2 d\ \ (S(u,u,.....), T(u,u,.....x_{2n-1})) + s^3 d\ \ (T(u,u,..., x_n), S(u,u,..., x_{2n-1}, x_{2n}))$$
$$+ .... + s^k d(T(u, x_{2n-1}, ..., x_{k\_2n-3}), S(x_{2n-1}, x_{2n}, ..., x_{k+2n-2}))$$
$$\leq sd(fu, y_{k+2n-1}) + s^2 \lambda\phi\{d(fu, fu), d(fu, fu), ...d(fu, fx_{2n-1})\}$$
$$+ s^3 \lambda\phi\{d(fu, fu), d(fu, fu), ..., d(fu, fx_{2n-1}), d(fx_{2n-1}, fx_{2n})\} + ...$$
$$+ s^k \lambda\phi\{d(fu, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{k+2n-3}, fx_{k+2n-2})\}$$
$$= sd(fu, y_{n+k}) + s^2 \lambda\phi\, d(0,0,....., d(fu, fx_{2n-1}) + s^3 \lambda\phi\, d(0,0,..., d(fu, fx_{2n-1}), d(fx_{2n-1}, fx_{2n})) + ...$$
$$+ s^{k+1} \lambda\phi\{d(fu, fx_{2n-1}), d(fx_{2n-1}, fx_{2n}), ..., d(fx_{k+2n-3}, fx_{k+2n-2})\}$$
$$<< \frac{sc}{s(k+1)} + s^2 \lambda\phi\left(\frac{c}{s^{k+1}\lambda(k+1)}\ , \frac{c}{s^{k+1}\lambda(k+1)}, ...\frac{c}{s^{k+1}\lambda(k+1)}\right)$$
$$+ s^3 \lambda\phi\left(\frac{c}{s^{k+1}\lambda(k+1)}\ , \frac{c}{s^{k+1}\lambda(k+1)}, ...\frac{c}{s^{k+1}\lambda(k+1)}\right) + ...$$
$$+ s^{k+1} \lambda\phi\left(\frac{c}{s^{k+1}\lambda(k+1)}\ , \frac{c}{s^{k+1}\lambda(k+1)}, ...\frac{c}{s^{k+1}\lambda(k+1)}\right)$$
$$<< \frac{c}{k+1} + \lambda\frac{c}{\lambda(k+1)} + ... + \lambda\frac{c}{\lambda(k+1)} \ = c$$

If k is hold, then for all $n \geq N_2$

$$d(fu, S(u,u,.......)) \leq s[d(fu, y_{k+2n-1}) + d(y_{k+2n-1}, S(u,u,........))]$$
$$= s[d(fu, y_{k+2n-1}) + d(S(x_{2n-1}, x_{2n}, ..., x_{k+2n-2}), S(u,u,........))]$$
$$\leq sd(fu, y_{k+2n-1}) + s^2 d\ \ (S(u,u,.....), T(u,u,.....x_{2n-1})) + s^3 d\ \ (T(u,u,..., x_n), S(u,u,..., x_{2n-1}, x_{2n}))$$
$$+ s^{k-1} d(T(u,u, x_{2n-1}, ..., x_{k-2n-4}), S(u, x_{2n-1}, x_{2n}, ..., x_{k+2n-3}))$$
$$+ .... + s^k d(S(u, x_{2n-1}, ..., x_{k-2n-3}), S(x_{2n-1}, x_{2n}, ..., x_{k+2n-2}))$$

Proceeding as above we get $d(fu, S(u,u,.........u) << c$.

Thus $d(fu, S(u,u,.........u) << \frac{c}{m}$ for all $m \geq 1$.

So, $\frac{c}{m} - d(fu, S(u,u,.........u) \in P$ for all $m \geq 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$ and P is closed,

$-d(fu, S(u,u,.........u) << \frac{c}{m}$ but $P \cap (-P) = |0|$. Therefore $d(fu, S(u,u,.........u) = 0$.

Thus $fu = S(u,u,.........u)$.

For eq. [1.2] we have

$fu = T(u,u,.........u)$, *i.e.,* $C(f, T, S) \neq 0$.

Theorem 2.Let (X, d) be a complete cone b-metric space with cone P and let T : X → X be a self-mapping of X such that d(Tx, Ty) ≤ α d(x, y) for all x, y ∈X, where α ≥ 0 is a constant. Then T has a unique fixed point.

Proof. Let x0 ∈ X. We set

x1 = Tx^0, $x^2 = Tx^1 = T^2x^0$, . . . , $x^{n+1} = Tx^n = T_{n+1}x^0$, . . .

We have

d(xn+1, xn) = d(Txn, Txn−1) ≤ _d(xn, xn−1) ≤ $\alpha 2$

≤ d(xn−1, xn−2) ≤ · · · ≤ _nd(x1, x0).

For n > m,

d(xn, xm) ≤ K(d(xn, xn−1) + d(xn−1, xn−2)) ≤ Kn(α n−1 + α n−2 + . . .) α d(x_1, x_0) ≤$\frac{\propto^m}{1-\propto}$ We get k $\|d(xn, xm)\| \leq \left\|\frac{\propto^m}{1-\propto}\right\| \|d(x1, x0\|$.

This implies d(xn, xm)→ 0(n,m → ∞). Hence {xn} is a Cauchy sequence. Since X is complete, therefore there is u ∈X such that xn → 0 as n → ∞. Since d(Tu, u) ≤ K(d(Txn, Tu) + d(Txn, u) ≤ K(_d(xn, u + d(x_{n+1}, u)), this implies

k d(Tu, u) k≤ K(K k d(xn, u) k + k d(xn+1, u) k) → 0.

Hence k d(Tu, u) = 0. This implies Tu = u. Thus u is a fixed point of T.

To prove the uniqueness: Let v be another fixed point of T, then d(Tu, Tv) ≤ Kd(u, v). Hence k d(u, v) = 0 k∈ u = v. Thus T has a unique fixed point.

Corollary 1. Let (X, d) be a complete cone b-metric space for c ∈ E with $_0$ ∈ c and x_0 ∈ X, set B(x_0, c) = x ∈ X : d(x_0, x) ≤ c. Suppose the mapping T : X → X be a self-mapping of X such that d(Tx, Ty) ≤ α d(x, y), for all x, y ∈ B(x_0, c), where α ∈ [0, 1] is a constant and d(Tx_0, x_0) ≤ (1 α)c.

Then T has a unique fixed point in B(x_0, c).

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