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# Legendre Multiwavelet Method for the Solution of Linear Fractional Time Delay Systems

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**ABSTRACT**

This research article provides the numerical technique for the solution of fractional time delay systems using Legendre multiwavelet method. The fundamental properties of Legendre multiwavelets along with collocation method are converting the problem of delay system into system of algebraic equations which can be solved with a suitable tool. The Lyapunov stability theorem is added for the fractional system with delay. Numerical examples are included to prove the straightforwardness, efficiency and wide range applicability of the present technique.

**Keywords:** Legendre Multiwavelet; Collocation method; Time delay system; Convergence.

## 1. INTRODUCTION

Fractional calculus is a generalization of the concept of ordinary differentiation and integration of an arbitrary non-integer order. They mainly occur in control systems.In the universe, most of the physical phenomena occur in the fractional order form and as a matter of fact, the mathematical models in interdisciplinary fields of science and engineering are containing fractional order terms.Therefore fractional calclulus became a significant field from last decade. The fractional models provide the excellent description of the history of the considered process and transmissible characteristics of the various phenomenon.It is because of all these reasons the fractional calculus is providing a better solution for the real world complicated models than integer calculus by its efficient properties.

Consider the following fractional system having time delay $\tau$ as

$${}_{t_0}^{c}D_t^{\alpha} y(x) = A(x)y(x) + \sum_{i=1}^{n} B_i(x)y(x-\tau_i) + K(x)R(x), \ x \in [a,b] \quad (1)$$

$$y(0) = y_0, \quad y(x) = \beta(x), x < 0 \quad (2)$$

Here ${}_{t_0}^{c}D_{t_f}^{\alpha} y(x)$ is the Caputo fractional derivative of the function $y$ and $\beta(x)$ is the given condition. Also, $y(x) \in R^n$, $R(x) \in R^m$, $K(x), B_i(x), A(x)$ are well defined dimension matrices with constant vectors $y(x)$ and $R(x)$.

In the literature survey, we found that the many scientists and researchers have studied on the fractional calculus. As, the accuracy is an important factor for the solution of fractional systems with time delay, the few number of numerical techniques were developed in the pasture of time. Research articles which are available are on existence and uniqueness of the fractional delay systems. Stability of the time delay systems have studied in[1]. Chen L *et al*. studied the finite time stability of fractional order linear and nonlinear delayed systems[2,3]. Jian-Bing *et al*. have given the Lyapunov stability theorem about fractional system without and with delay[4]. In this paper, we present a numerical technique for the solution of fractional delay systems by using Legendre multiwavelets. The properties of Legendre multiwavelets along with collocation method are used to convert the linear systems with time delay into system of algebraic equations with unknown coefficients then we find the value of unknown coefficients and by substituting back to find an approximate solution $y(x)$ of the equations (1) and (2). Also, we have used the Caputo fractional derivative as it has better physical interpretations. Since the derivative of a constant is zero or in formulation of real world problems, the initial conditions are formulated in terms of integer order derivatives which can be treated as initial situation and initial rate of the systems.

Wavelet theory is a new and emerging area in the field of applied science and engineering. It has wide range of applications in many branches. Wavelets are useful in signal analysis waveform-representation, time- frequency analysis, image processing, fast algorithms, etc. The wavelets are highly useful in numerical analysis due to the fast converging properties.

The remaining part of this article is organised as follows: The basics of fractional calculus and Legendre wavelets are briefly narrated in section 2. In section 3, the method of solution for linear fractional time delay systems is given. Some bench-mark examples are included to reveal the efficiency and wide applicability of the present method in section 4. Finally, the conclusions are drawn in section 5.

## 2. PRELIMINARIES OF FRACTIONAL CALCULUS AND LEGENDRE MULTIWAVELETS

In this section, we recollect some basic definitions and results of fractional calculus theory and Legendre wavelets which are used in this paper.

### 2.1 Fractional Calculus

Definition 1: The fractional integral operator of Riemann-Liouville of order $\mu > 0$ is defined as

$$I^{\mu} f(x) = \frac{1}{\Gamma\mu} \int_0^x (x-s)^{\mu-1} f(s) ds, x > 0 \tag{3}$$

The real function $f(x), x > 0$ of the equation (3) satisfies the following properties:

(i) For $\mu > 0$, $I^{\mu}\left(af(x) + bg(x)\right) = aI^{\mu} f(x) + bI^{\mu} g(x)$ , where $a$ and $b$ are the constants.

(ii) For $n > -1$, $I^{\mu} x^n = \frac{\Gamma(n+1)}{\Gamma(n+\mu+1)} x^{n+\mu}$.

Definition 2: The fractional derivative of $f(x), x > 0$ in the Caputo sense is defined as

$${}_0^C D_x^{\mu} f(x) = \frac{1}{\Gamma n-\mu} \int_0^x (x-s)^{n-\mu-1} f^{n}(x) ds \tag{4}$$

where $x > 0, \mu \in R, \mu \in (n-1, n] \,\&\, n \in N$ .

The above equation (4) containing the real function $f(x), x > 0$ has the following properties:

(i) $D^{\mu}\left(I^{\mu} f(x)\right) = f(x)$.

(ii) $I^{\mu}\left(D^{\mu} f(x)\right) = f(x) - \sum_{k=0}^{n-1} f^{(k)} \frac{t^k}{k!}$ for $k > 0$.

### 2.2 Legendre multiwavelets

Wavelets are a family of functions made up from the dilation and translation of one function called mother wavelet. If the dilation parameter $a$ and the translation parameter $b$ change continuously, we have the following wavelet family as

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \left(\frac{t-b}{a}\right), \ a, b \in R \,\&\, a \neq 0 \tag{5}$$

If, we limit the $a$ and $b$ parameters to distinguish values as $a = a_0^m$, $b = nb_0 a_0^m, a_0 > 1 \,\&\, b_0 > 0$ and $m, n$ are the positive integers. Then we have the family of discrete wavelets as follows:

$$\psi_{m,n}(t) = |a_0|^{\frac{m}{2}} \psi\left(a_0^m t - nb_0\right)$$

where $\psi_{m,n}(t)$ form the wavelet basis for $L^2(R)$ and $a_0 = 2\ \&\ b_0 = 1$ then $\psi_{m,n}(t)$ forms an orthonormal basis.

The Legendre multiwavelets have four arguments $\psi_{n,m}(t) = \psi(k,n,m,t)$ where $n = 0,1,2,\dots,2^k - 1$, $k \geq 1, m$ is the order of Legendre polynomials and t is the time defined on $0 \leq t \leq T$.

Legendre multiwavelets functions on the interval $[0,T)$ are defined as

$$\psi_{n,m}(t) = \begin{cases} UP_m\left(\dfrac{2^k t}{T} - n\right), \dfrac{nT}{2^k} \leq t < \dfrac{(n+1)T}{2^k} \\ \qquad 0, \qquad \text{otherwise} \end{cases} \tag{6}$$

Where $U = \sqrt{2m+1}.\dfrac{2^{k/2}}{\sqrt{T}}$ and $\sqrt{2m+1}$ is the coefficient for orthonormality, dilation parameter is $a = 2^{-k}T$ & translation parameter $b = n2^{-k}T$. Also, $P_m(t)$ are the well known shifted Legendre polynomials having order $m$ and are defined in the interval $[0,1]$. One can determine the using the following recurrence formula

$$P_{m+1}(t) = \frac{(2m+1)}{(m+1)}(2t-1)P_m(t) - \frac{(m)}{(m+1)}P_{m-1}(t)\ , m = 1,2,3,\dots \tag{7}$$

$$\text{with } P_0(t) = 1, P_1(t) = 2t - 1$$

### 2.3 Function approximation

The following results give an upper bound for the error that occurs in the approximation of functions using Legendre polynomials.

Theorem 1: Let $f$ be a continuous function which is differentiable $n$ times and

$$g_j = \frac{\int\limits_{x=0}^{1} p_j(x)\, f(x)\, dx}{\int\limits_{x=0}^{1} p_j(x)\, p_j(x)\, dx} \quad \text{then,} \quad f - \sum_{j=0}^{N} g_j p_j L_2 \leq \frac{c}{N^n} \sum_{k=0}^{n} f^{(k)} L^2, k \geq 0$$

Proof: Let a function $f(x)$ defined over the interval $[0,T]$ can be expressed as

$$f(x) \approx \sum_{i=0}^{k-1} \sum_{j=0}^{\infty} c_{i,j} \psi_{i,j} \tag{8}$$

Where $k \geq 0$. If the infinite series is truncated in the above equation (8) and is re-expressed as

$$f(x) \cong C^T \psi(x) = \sum_{i=0}^{k-1} \sum_{j=0}^{M} c_{i,j} \psi_{i,j}$$

Where $c_{i,j}$ coefficient is associated with $\psi_{i,j}$ and can be calculated by using the following equation

$$C_{i,j} = \frac{\int_0^T \psi_{i,j}(x) f(x) dx}{\int_0^T \psi_{i,j}(x) \psi_{i,j}(x) dx}, i = 0,1,2,...,k-1 \,\&\, j = 0,1,2,...,M \tag{9}$$

and $C^T = \left[ c_{0,0}, c_{0,1}, ..., c_{0,M}, c_{1,0}, c_{1,1}, ..., c_{1,M}, c_{(k-1),0}, c_{(k-1),1}, ..., c_{(k-1),M} \right]$ (10)

$$\psi(x)^T = \left[ \psi_{0,0}, \psi_{0,1}, ..., \psi_{0,M}, \psi_{1,0}, \psi_{1,1}, ..., \psi_{1,M}, ...., \psi_{(k-1),0}, \psi_{(k-1),1}, ..., \psi_{(k-1),M} \right] \tag{11}$$

We have two degrees of freedom and that increases the accuracy of the method by using Legendre multiwavelet basis. The two parameters, argument k and M that associated to the number of elements in every subinterval $\left[ \frac{rT}{k}, \frac{(r+1)T}{k} \right]$ of Legendre basis.

The following Lemma gives proper inequality to estimate the error of an upper bound.

Lemma 1: If the function $f : [0,T] \to R$ is continuously differentiable $(M+1)$ times and $f \in C^{M+1}[0,T]$ then $f$ can be approximated by $C^T \psi$ as follows:

$$\left| f - C^T \psi \right|_{L^2} \leq \frac{T^{\frac{2M-3}{2}}}{(M+1)! \left( \sqrt{2M+3k} \right)^{\frac{2M+3}{2}}} \sup_{x \in [0,T]} \left| f^{(M+1)}(x) \right| \tag{12}$$

Proof is obvious by using the Taylor expansion of $f$.

## 3. SOLVING TIME DELAY SYSTEM HAVING FRACTIONAL ORDER DERIVATIVE

Let us consider the fractional system having time delay $\tau$ as from equations (1) and (2) in the following form as:

$${}_{t_0}^{c}D_t^{\alpha} y(x) = A(x) y(x) + \sum_{i=1}^{n} B_i(x) y(x - \tau_i) + K(x) R(x), \; x \in [a,b] \tag{13}$$

$$y(0)=y_0, \quad y(x)=\beta(x),\ x<0 \tag{14}$$

where $y_0=\left(y_{0_1},y_{0_2},...,y_{0_n}\right)$ (15)

We find the dilation parameter $k$ such that $\tau_j=\frac{jT}{k}$, $j=1,2,3,...,L$ and approximate $y(x)=\left(y_1(x),y_2(x),...,y_n(x)\right)$ in the truncated series form by using Legendre multiwavelets as defined in equation (6) as follows:

$$y_r(x)\cong\sum_{i=0}^{k-1}\sum_{j=0}^{M}C_{r,i,j}\psi_{i,j}(x), r\in[1,n] \tag{16}$$

and also, we consider $y_r(x)=\sum_{i=0}^{k-1}\sum_{j=0}^{M}C_{r,i,j}\psi_{i,j}(x)$ with

$$\bar{y}(x)=\left(\bar{y}_1(x),\bar{y}_2(x),...,\bar{y}_n(x)\right) \ \&\ \bar{y}(x)=\phi(x) \text{ for } i\in[0,k-1]. \tag{17}$$

$$\text{Let },\psi_{i,j}(x)=\begin{cases}UP_m\left(\frac{2^k x}{T}-n\right), \frac{nT}{2^k}\le x<\frac{(n+1)T}{2^k}\\ 0, \quad \text{otherwise}\end{cases} \tag{18}$$

Now, we rewrite the system (13) as

$$V(x)={}_0^cD_x^\alpha\,\bar{y}(x)-A(x)y(x)-\sum_{i=1}^{n}B_i(x)y(x-\tau_i)-K(x)R(x), x\in[0,T] \tag{19}$$

To find the values of the coefficients $C_{r,i,j}$, $i\in[0,k-1], j\in[1,M]\ \&\ r\in[1,n]$. We build an algebraic system of equations containing $nk(M+1)$ unknowns and $nk(M+1)$ equations in such a way that the reminder $V(x)$ nodes $\frac{jT}{k}+\frac{iT}{k(M+1)}$ for $i=1,2,3,...,M\ \&\ j=0,1,2,...,k-1$ give proper initial conditions of the system (13) and also give continuous solution for the given problem.

$$\sum_{j=0}^{M}C_{r,o,j}\psi_{0,j}(0)=y_{0_r}, r\in[1,n] \tag{20}$$

$$\sum_{j=0}^{M}C_{r,i-1,j}\psi_{i-1,j}\left(\frac{iT}{k}\right)=\sum_{j=0}^{M}C_{r,i,j}\psi_{i,j}\left(\frac{iT}{k}\right) \text{ for } i=1,2,3,...,M\ \&\ j=0,1,2,...,k-1 \tag{21}$$

and $$V\left(\frac{jT}{k}+\frac{iT}{k(M+1)}\right)=0 \text{ for } i=1,2,3,...,M \;\&\; j=0,1,2,...,k-1 \tag{22}$$

On solving the system of equations from (20) to (22), we find the values of the coefficients of the series (16). The equation (20) provides the initial condition of the system (13) & (14). Also, the equation (21) gives continuous approximate solution of the system.

## 4. NUMERICAL EXPERIMENTS

In this section, the proposed method in the above section is implemented for solving some standard examples to illustrate the efficiency of the method.

Example 1. Consider the following linear fractional time delay system[18]

$${}^{c}D_{0,t}^{0.5}y(t)=\begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}+\begin{pmatrix}0 & 0\\ 3 & 4\end{pmatrix}\begin{pmatrix}y_1(t-\tau)\\ y_2(t-\tau)\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}u(t) \tag{23}$$

With initial condition $y(t)=\varphi(t)=0,-0.1\le t\le 0.1.$

where $u(t)=\left[u_1(t),u_2(t)\right]^T$, $\delta=0.1,\in=100 \;\&\; \tau=0.1$. This equation (23) can be solved by using the procedure as described in the previous section. It can be easily verify that $\|A_0\|=2 \;\&\; \|A_1\|=5$ and $\|\phi\|=\delta$. Using the inequality $r_T(\tau)E_\alpha\left(V_0T^\alpha\right)\le\frac{\in}{\delta}$ [18] with $\alpha=0.5$, $\gamma_u=1$ the computed estimated time f the finite-time stability of system and is T=0.2 while by checking the estimated time of the finite-time stability in the[3] is T=0.1. Therefore it is clear that the solution from this method converges to the exact solution faster.

Example 2. Consider the following linear fractional time delay system[18]

$${}^{c}D_{0,t}^{0.2}y(t)=\begin{pmatrix}0.2 & 0\\ 0 & 0.8\end{pmatrix}\begin{pmatrix}y_1(t-\tau)\\ y_2(t-\tau)\end{pmatrix} \tag{24}$$

with initial condition $y(t)=\varphi(t)=[0.1,0.2]^T,-\tau\le t\le 0.$

Where, $\delta=0.31,\in=1.29\;\; \tau=0.2 \;\&\; \alpha=0.2$. We apply the same procedure for this equation (24) and compute the values $\|A_0\|=0.8$ and $\|\phi\|\le\delta$ .By applying the inequality $l_T(\tau)E_\alpha\left(V_0T^\alpha\right)\le\in$ [18], we can easily obtain the finite-time stability of system[3] is T=0.8. It is very close to the exact solution of the system (24).

## 5. CONCLUSION

In this paper, fractional order time delay systems have been solved by applying the properties of Legendre multiwavelets along with the collocation method successively. By using this technique, we reduce the given delay system into system of algebraic equations which can be solved easily. The illustrative examples are included to prove the efficiency, validity and applicability of the technique. It can be observed from the obtained numerical solutions that only a small number of Legendre multiwavelets are sufficient to get more accurate solutions. Further, the same technique can be utilized for the solution of various kinds of systems.

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