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Super Graceful Labeling for Some Families of Fan Graphs

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ABSTRACT

Let G be a (p, q) graph. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3,..., p+q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a super graceful labeling. A graph G is called a super graceful graph if it admits a super graceful labeling.

In this paper, we study the super gracefulness of Fan graph F_n , Double Fan graph DF_n and SF(m, n).

Keywords: Graceful labeling, Super graceful labeling, Fan graph.

1. INTRODUCTION

All graphs in this paper are finite and undirected. The symbol $|V(G)|$ and $|E(G)|$ denotes the number of vertices and edges in a graph G. A graph labeling is an assignment of integers to the vertices or edges. An injective function $f : V(G) \rightarrow \{0, 1, 2, 3,..., q\}$ is called a graceful labeling of G if all the edge labels of G given by $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ are distinct[7].

In[4] and[5], the authors introduced the concept of super graceful labeling. Let G be a (p, q) graph. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, 3,..., p+q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a super graceful labeling. A graph G is called a super graceful graph if it admits a super graceful labeling[4].

In[3], the authors conjectured that all trees are super graceful.

In this paper we study the super gracefulness of Fan graph F_n, Double Fan graph DF_n, SF(m, n) and the graph G obtained by identifying central vertex of S_n to a central vertex of F_n and DF_n.

Definition 1.1 A star graph S_n is the tree of order n with maximum diameter 2 with n-1 leaves. (Alternatively, some authors define S_n is the complete bipartite graph $K_{1,n}$: a tree with one internal node and n leaves.)

Definition 1.2 A Fan graph F_n is defined as the graph $K_1 + P_n$, where K_1 is the empty graph on one vertex and P_n, $n \geq 2$ is the path graph on n vertices.

Definition 1.3 The double Fan graph DF_n consists of two fan graph that have a common path. In other words, $DF_n = P_n + \overline{K_2}$, $n \geq 2$

Definition 1.4 The graph SF(m, n) is a graph obtained by identifying a central vertex of star graph S_m, $m \geq 3$ with a end vertex of path in Fan graph $F_n = K_1 + P_n$, $n \geq 2$.

2. SUPER GRACEFUL FAN GRAPHS

Theorem 2.1 F_n is a super graceful graph.

Proof: Let $G = F_n$

Here $V(G) = \{v_1, v_2, ..., v_n, u\}$ and

$E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{uv_i / 1 \leq i \leq n\}$

Therefore, $|V(G)| = n+1$, $|E(G)| = 2n-1$ and $|V(G) \cup E(G)| = n+1+2n-1 = 3n$

Case (i) n is even

Define $f : V(G) \cup E(G) \to \{1, 2, 3, ..., 3n\}$ as follows.

f(u) = n

$$f(v_{2i+1}) = 3n - i, \quad 0 \leq i \leq \frac{n}{2} - 1$$

$$f(v_{2i}) = 2n + i, \quad 1 \leq i \leq \frac{n}{2}$$

Therefore, the set of vertex labels of the vertex set is

$$\{n\} \cup \left\{3n, 3n-1, 3n-2, ..., \frac{5n+2}{2}\right\} \cup \left\{2n+1, 2n+2, ..., \frac{5n}{2}\right\}$$

The induced edge labels are

$$f^*(v_{2i+1}u) = 2n - i, \quad 0 \leq i \leq \frac{n}{2} - 1$$

$f^*(v_{2i}u) = n + i\ ,\ 1 \le i \le \frac{n}{2}$

$f^*(v_i v_{i+1}) = \{\ n - i\ \ /1 \le i \le n - 1\}$

Therefore, the set of edge labels of the edge set is

$\left\{2n\ , 2n - 1, ..., \frac{3n+2}{2}\right\} \cup \left\{n+1, n+2, ..., \frac{3n}{2}\right\} \cup \{\ n-1, n-2, ..., 2, 1\ \}$

Case (ii) n is odd

Define $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., 3n\}$ as follows.

f(u) = n

$f(v_{2i+1}) = 3n - i\ ,\ 0 \le i \le \frac{n-1}{2}$

$f(v_{2i}) = 2n + i\ ,\ \ 1 \le i \le \frac{n-1}{2}$

Therefore, the set of vertex labels of the vertex set is

$\{\ n\ \} \cup \left\{3n, 3n - 1, 3n - 2, ..., \frac{5n+1}{2}\right\} \cup \left\{2n+1, 2n+2, ..., \frac{5n-1}{2}\right\}$

The induced edge labels are

$f^*(v_{2i+1}u) = 2n - i\ ,\ 0 \le i \le \frac{n-1}{2}$

$f^*(v_{2i}u) = n + i\ ,\ 1 \le i \le \frac{n-1}{2}$

$f^*(v_i v_{i+1}) = \{\ n - i\ \ /1 \le i \le n - 1\}$

Therefore, the set of edge labels of the edge set is

$\left\{2n\ , 2n - 1, ..., \frac{3n+1}{2}\right\} \cup \left\{n+1, n+2, ..., \frac{3n-1}{2}\right\} \cup \{\ n-1, n-2, ..., 2, 1\ \}$

In both case, we observe that all the vertex labels and edge labels are distinct and their union is $\{1, 2, 3, ..., 3n\}$.

Therefore, f is a super graceful labeling and hence, F_n is a super graceful graph.

Example 2.1.

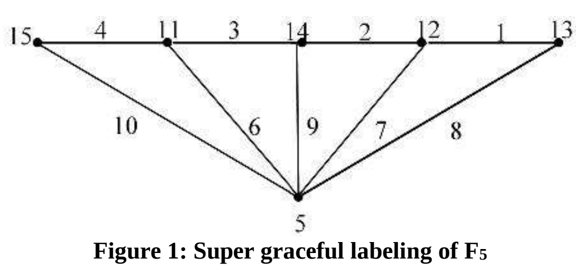

Figure 1: Super graceful labeling of F_5

Theorem 2.2 DF_n is a super graceful graph.

Proof: Let G = DF_n

Here $V(G) = \{v_1, v_2, ..., v_n, u, w\}$ and

$E(G) = \{v_i v_{i+1} / 1 \le i \le n-1\} \cup \{uv_i / 1 \le i \le n\} \cup \{wv_i / 1 \le i \le n\}$

Therefore, $|V(G)| = n+2$, $|E(G)| = 3n-1$ and $|V(G) \cup E(G)| = n+2+3n-1 = 4n+1$

Case (i) n is even

Define $f : V(G) \cup E(G) \to \{1, 2, 3, ..., 4n+1\}$ as follows.

f(u) = n

$f(v_{2i+1}) = 4n+1-i \;,\; 0 \le i \le \frac{n}{2}-1$

$f(v_{2i}) = 3n+1+i \;,\; 1 \le i \le \frac{n}{2}$

f(w) = 2n +1

Therefore, the set of vertex labels of the vertex set is

$\{n\} \cup \left\{4n+1, 4n, 4n-1, ..., \frac{7n+4}{2}\right\} \cup \left\{3n+2, 3n+3, ..., \frac{7n+2}{2}\right\} \cup \{2n+1\}$

The induced edge labels are

$f^*(v_{2i+1}u) = 3n+1-i \;,\; 0 \le i \le \frac{n}{2}-1$

$f^*(v_{2i}u) = 2n+1+i \;,\; 1 \le i \le \frac{n}{2}$

$f^*(v_i v_{i+1}) = \{n-i \;/ 1 \le i \le n-1\}$

$f^*(v_{2i+1}w) = 2n-i \;,\; 0 \le i \le \frac{n}{2}-1$

$f^*(v_{2i}w) = n+i \;,\; 1 \le i \le \frac{n}{2}$

Therefore, the set of edge labels of the edge set is

$\left\{3n+1, 3n, ..., \frac{5n+4}{2}\right\} \cup \left\{2n+2, 2n+3, ..., \frac{5n+2}{2}\right\} \cup \{n-1, n-2, ..., 2, 1\} \cup \left\{2n, 2n-1, ..., \frac{3n+2}{2}\right\} \cup \left\{n+1, n+2, ..., \frac{3n}{2}\right\}$

Case (ii) n is odd

Define $f : V(G) \cup E(G) \to \{1, 2, 3, ..., 4n+1\}$ as follows.

f(u) = n

$f(v_{2i+1}) = 4n + 1 - i \ , \ 0 \le i \le \frac{n-1}{2}$

$f(v_{2i}) = 3n + 1 + i \ , \quad 1 \le i \le \frac{n-1}{2}$

$f(w) = 2n + 1$

Therefore, the set of vertex labels of the vertex set is

$$\{ n \} \cup \left\{ 4n+1, 4n, 4n-1, ..., \frac{7n+3}{2} \right\} \cup \left\{ 3n+2, 3n+3, ..., \frac{7n+1}{2} \right\} \cup \{ 2n+1 \}$$

The induced edge labels are

$f^*(v_{2i+1}u) = 3n + 1 - i \ , \ 0 \le i \le \frac{n-1}{2}$

$f^*(v_{2i}u) = 2n + 1 + i \ , \ 1 \le i \le \frac{n-1}{2}$

$f^*(v_i v_{i+1}) = \{ n - i \ / 1 \le i \le n-1 \}$

$f^*(v_{2i+1}w) = 2n - i \ , \ 0 \le i \le \frac{n-1}{2}$

$f^*(v_{2i}w) = n + i \ , \ 1 \le i \le \frac{n-1}{2}$

Therefore, the set of edge labels of the edge set is

$$\left\{ 3n+1, 3n, ..., \frac{5n+3}{2} \right\} \cup \left\{ 2n+2, 2n+3, ..., \frac{5n+1}{2} \right\} \cup \{ n-1, n-2, ..., 2, 1 \} \cup \left\{ 2n, 2n-1, ..., \frac{3n+1}{2} \right\} \cup$$
$$\left\{ n+1, n+2, ..., \frac{3n-1}{2} \right\}$$

In both case, we observe that all the vertex labels and edge labels are distinct and their union is $\{1, 2, 3, ..., 4n+1\}$.

Therefore, f is a super graceful labeling and hence, DF_n is a super graceful graph.

Example 2.2.

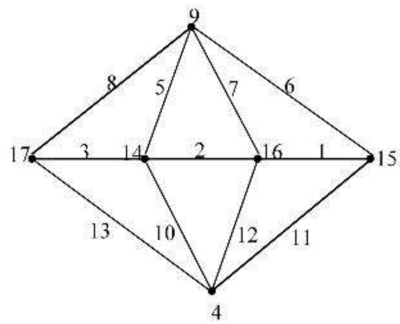

Figure 2: Super graceful labeling of DF_4

Theorem 2.3 SF(m,n) is a super graceful graph.

Proof: Let G = SF(m,n)
Here $V(G)=\{v_1, v_2,..., v_n, u, u_1, u_2,..., u_{m-1}\}$ and
$E(G)=\{v_i v_{i+1} / 1\le i\le n-1\}\cup\{uv_i / 1\le i\le n\}\cup\{v_1u_i / 1\le i\le m-1\}$
Therefore, $|V(G)|=m+n$, $|E(G)|=2n+m-2$ and $|V(G)\cup E(G)|=2m+3n-2$

Case (i) n is even

Define $f: V(G)\cup E(G)\to\{1, 2, 3,..., 2m+3n-2\}$ as follows.
f(u) = n+m-1

$$f(v_{2i+1}) = 2m+3n-2-i\ ,\ 0\le i\le \frac{n}{2}-1$$

$$f(v_{2i}) = 2(m+n-1)+i\ ,\ 1\le i\le \frac{n}{2}$$

$$f(u_i) = n-1+i\ ,\ 1\le i\le m-1$$

Therefore, the set of vertex labels of the vertex set is

$$\{n+m-1\}\cup\left\{2m+3n-2, 2m+3n-3, ..., \frac{4m+5n-2}{2}\right\}\cup$$

$$\left\{2m+2n-1, 2m+2n, ..., \frac{4m+5n-4}{2}\right\}\cup\{n, n+1, ..., n+m-2\}$$

The induced edge labels are

$$f^*(v_{2i+1}u) = m+2n-1-i\ ,\ 0\le i\le \frac{n}{2}-1$$

$$f^*(v_{2i}u) = m+n-1+i\ ,\ 1\le i\le \frac{n}{2}$$

$$f^*(v_i v_{i+1}) = \{n-i\ / 1\le i\le n-1\}$$

$$f^*(v_1 u_i) = 2m+2n-1-i\ ,\ 1\le i\le m-1$$

Therefore, the set of edge labels of the edge set is

$$\left\{m+2n-1, m+2n-2, ..., \frac{2m+3n}{2}\right\}\cup\left\{m+n, m+n+1, ..., \frac{2m+3n-2}{2}\right\}\cup$$

$$\{n-1, n-2, ..., 2, 1\}\cup\{2m+2n-2, 2m+2n-3, ..., m+2n\}$$

Case (ii) n is odd

Define $f: V(G)\cup E(G)\to\{1, 2, 3,..., 2m+3n-2\}$ as follows.
f(u) = n+m-1

$$f(v_{2i+1}) = 2m + 3n - 2 - i \ , \ 0 \le i \le \frac{n-1}{2}$$

$$f(v_{2i}) = 2(m + n - 1) + i \ , \quad 1 \le i \le \frac{n-1}{2}$$

$$f(u_i) = n - 1 + i \ , \quad 1 \le i \le m - 1$$

Therefore, the set of vertex labels of the vertex set is

$$\{n + m - 1\} \cup \left\{2m + 3n - 2, 2m + 3n - 3, ..., \frac{4m + 5n - 3}{2}\right\} \cup \left\{2m + 2n - 1, 2m + 2n, ..., \frac{4m + 5n - 5}{2}\right\} \cup \{n, n + 1, ..., n + m - 2\}$$

The induced edge labels are

$$f^*(v_{2i+1}u) = m + 2n - 1 - i \ , \ 0 \le i \le \frac{n-1}{2}$$

$$f^*(v_{2i}u) = m + n - 1 + i \ , \ 1 \le i \le \frac{n-1}{2}$$

$$f^*(v_i v_{i+1}) = \{n - i \ / 1 \le i \le n - 1\}$$

$$f^*(v_1 u_i) = 2m + 2n - 1 - i \ , \ 1 \le i \le m - 1$$

Therefore, the set of edge labels of the edge set is

$$\left\{m + 2n - 1, m + 2n - 2, ..., \frac{2m + 3n - 1}{2}\right\} \cup \left\{m + n, m + n + 1, ..., \frac{2m + 3n - 3}{2}\right\} \cup \{n - 1, n - 2, ..., 2, 1\} \cup \{2m + 2n - 2, 2m + 2n - 3, ..., m + 2n\}$$

In both case, we observe that all the vertex labels and edge labels are distinct and their union is $\{1, 2, 3, ..., 2m + 3n - 2\}$.

Therefore, f is a super graceful labeling and hence, SF(m, n) is a super graceful graph.

Example 2.3.

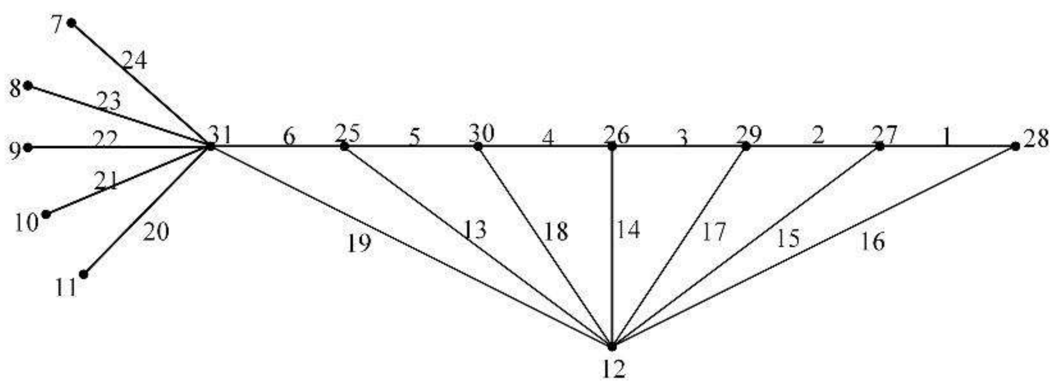

Figure 3: Super graceful labeling of SF(6, 7)

Theorem 2.4 Let G be the graph obtained by identifying centre vertex of star S_{n+1} to the centre vertex of $F_n = K_1 + P_n$. Then G is a super graceful graph.

Proof: Here $V(G) = \{v_1, v_2, ..., v_n, u, u_1, u_2, ..., u_n\}$ and

$E(G) = \{v_i v_{i+1} / 1 \le i \le n-1\} \cup \{uv_i / 1 \le i \le n\} \cup \{uu_i / 1 \le i \le n\}$

Therefore, $|V(G)| = 2n+1$, $|E(G)| = 3n-1$ and $|V(G) \cup E(G)| = 2n+1+3n-1 = 5n$

Case (i) n is even

Define $f : V(G) \cup E(G) \to \{1, 2, 3, ..., 5n\}$ as follows.

f(u) = n

$$f(v_{2i+1}) = 5n - i \ , \ 0 \le i \le \frac{n}{2} - 1$$

$$f(v_{2i}) = 4n + i \ , \ 1 \le i \le \frac{n}{2}$$

$$f(u_i) = 2n + i \ , \ 1 \le i \le n$$

Therefore, the set of vertex labels of the vertex set is

$$\{n\} \cup \left\{5n, 5n-1, ..., \frac{9n+2}{2}\right\} \cup \left\{4n+1, 4n+2, ..., \frac{9n}{2}\right\} \cup \{2n+1, 2n+2, ..., 3n\}$$

The induced edge labels are

$$f^*(v_{2i+1}u) = 4n - i \ , \ 0 \le i \le \frac{n}{2} - 1$$

$$f^*(v_{2i}u) = 3n + i \ , \ 1 \le i \le \frac{n}{2}$$

$$f^*(v_i v_{i+1}) = \{n - i \ / 1 \le i \le n-1\}$$

$$f^*(u_i u) = n + i \ , \ 1 \le i \le n$$

Therefore, the set of edge labels of the edge set is

$$\left\{4n, 4n-1, ..., \frac{7n+2}{2}\right\} \cup \left\{3n+1, 3n+2, ..., \frac{7n}{2}\right\} \cup \{n-1, n-2, ..., 2, 1\} \cup \{n+1, n+2, ..., 2n\}$$

Case (ii) n is odd

Define $f : V(G) \cup E(G) \to \{1, 2, 3, ..., 5n\}$ as follows.

f(u) = n

$$f(v_{2i+1}) = 5n - i \ , \ 0 \le i \le \frac{n-1}{2}$$

$$f(v_{2i}) = 4n + i \ , \ 1 \le i \le \frac{n-1}{2}$$

$$f(u_i) = 2n + i \ , \ 1 \le i \le n$$

Therefore, the set of vertex labels of the vertex set is

$$\{ n \} \cup \left\{ 5n , 5n - 1, ..., \frac{9n+1}{2} \right\} \cup \left\{ 4n + 1, 4n + 2, ..., \frac{9n-1}{2} \right\} \cup \{ 2n + 1, 2n + 2, ..., 3n \}$$

The induced edge labels are

$$f^*(v_{2i+1}u) = 4n - i \ , \ 0 \le i \le \frac{n-1}{2}$$

$$f^*(v_{2i}u) = 3n + i \ , \ 1 \le i \le \frac{n-1}{2}$$

$$f^*(v_i v_{i+1}) = \{ n - i \ / 1 \le i \le n-1 \}$$

$$f^*(u_i u) = n + i \ , \ 1 \le i \le n$$

Therefore, the set of edge labels of the edge set is

$$\left\{ 4n , 4n - 1, ..., \frac{7n+1}{2} \right\} \cup \left\{ 3n + 1, 3n + 2, ..., \frac{7n-1}{2} \right\} \cup \{ n - 1, n - 2, ..., 2, 1 \} \cup \{ n + 1, n + 2, ..., 2n \}$$

In both case, we observe that all the vertex labels and edge labels are distinct and their union is $\{1, 2, 3, ..., 5n\}$.

Therefore, f is a super graceful labeling and hence, G is a super graceful graph.

Example 2.4. Super graceful labelings of identifying centre vertex of S_5 , F_4 is given in Figure 4.

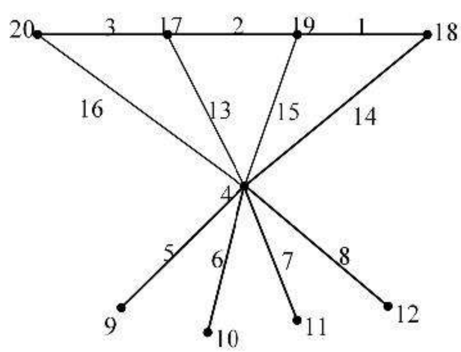

Figure 4

Theorem 2.5 Let G be the graph obtained by identifying centre vertex of star S_{n+1} to the centre vertex of $DF_n = \overline{K_2} + P_n$.. Then G is a super graceful graph.

Proof: Here $V(G) = \{v_1, v_2, ..., v_n, u, w, u_1, u_2, ..., u_n\}$ and

$$E(G) = \{v_i v_{i+1} \ / 1 \le i \le n-1\} \cup \{uv_i \ / 1 \le i \le n\} \cup \{wv_i \ / 1 \le i \le n\} \cup \{u_i u \ / 1 \le i \le n\}$$

Therefore, $|V(G)| = 2n + 2$, $|E(G)| = 4n - 1$ and $|V(G) \cup E(G)| = 2n + 2 + 4n - 1 = 6n + 1$

Case (i) n is even

Define $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., 6n+1\}$ as follows.

f(u) = n

$$f(v_{2i+1}) = 6n + 1 - i \ , \ 0 \leq i \leq \frac{n}{2} - 1$$

$$f(v_{2i}) = 5n + 1 + i \ , \ 1 \leq i \leq \frac{n}{2}$$

f(w) = 4n +1

$$f(u_i) = 3n + i \ , \ 1 \leq i \leq n$$

Therefore, the set of vertex labels of the vertex set is

$$\{ n \} \cup \left\{ 6n+1, 6n, ,..., \frac{11n+4}{2} \right\} \cup \left\{ 5n+2, 5n+3, ..., \frac{11n+2}{2} \right\} \cup \{ 4n+1 \} \cup \{ 3n+1, 3n+2, ..., 4n \}$$

The induced edge labels are

$$f^*(v_{2i+1}u) = 5n + 1 - i \ , \ 0 \leq i \leq \frac{n}{2} - 1$$

$$f^*(v_{2i}u) = 4n + 1 + i \ , \ 1 \leq i \leq \frac{n}{2}$$

$$f^*(v_i v_{i+1}) = \{ n - i \ / 1 \leq i \leq n - 1 \}$$

$$f^*(v_{2i+1}w) = 2n - i \ , \ 0 \leq i \leq \frac{n}{2} - 1$$

$$f^*(v_{2i}w) = n + i \ , \ 1 \leq i \leq \frac{n}{2}$$

$$f^*(u_i u) = 2n + i \ , \ 1 \leq i \leq n$$

Therefore, the set of edge labels of the edge set is

$$\left\{ 5n+1, 5n, ..., \frac{9n+4}{2} \right\} \cup \left\{ 4n+2, 4n+3, ..., \frac{9n+2}{2} \right\} \cup \{ n-1, n-2, ..., 2, 1 \} \cup \left\{ 2n, 2n-1, ..., \frac{3n+2}{2} \right\} \cup$$
$$\left\{ n+1, n+2, ..., \frac{3n}{2} \right\} \cup \{ 2n+1, 2n+2, ..., 3n \}$$

Case (ii) n is odd

Define $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., 6n+1\}$ as follows.

f(u) = n

$$f(v_{2i+1}) = 6n + 1 - i \ , \ 0 \leq i \leq \frac{n-1}{2}$$

$$f(v_{2i}) = 5n + 1 + i \ , \ 1 \leq i \leq \frac{n-1}{2}$$

$f(w) = 4n + 1$

$f(u_i) = 3n + i \ , \quad 1 \le i \le n$

Therefore, the set of vertex labels of the vertex set is

$$\{ n \} \cup \left\{ 6n+1, 6n, ,..., \frac{11n+3}{2} \right\} \cup \left\{ 5n+2, 5n+3,..., \frac{11n+1}{2} \right\} \cup \{ 4n+1 \} \cup \{ 3n+1, 3n+2,...,4n \}$$

The induced edge labels are

$$f^*(v_{2i+1}u) = 5n+1-i \ , \ 0 \le i \le \frac{n-1}{2}$$

$$f^*(v_{2i}u) = 4n+1+i \ , \ 1 \le i \le \frac{n-1}{2}$$

$$f^*(v_i v_{i+1}) = \{ n-i \ / 1 \le i \le n-1 \}$$

$$f^*(v_{2i+1}w) = 2n-i \ , \ 0 \le i \le \frac{n-1}{2}$$

$$f^*(v_{2i}w) = n+i \ , \ 1 \le i \le \frac{n-1}{2}$$

$$f^*(u_i u) = 2n+i \ , \ 1 \le i \le n$$

Therefore, the set of edge labels of the edge set is

$$\left\{ 5n+1, 5n,..., \frac{9n+3}{2} \right\} \cup \left\{ 4n+2, 4n+3,..., \frac{9n+1}{2} \right\} \cup \{ n-1, n-2,...,2,1 \} \cup \left\{ 2n, 2n-1,..., \frac{3n+1}{2} \right\} \cup$$

$$\left\{ n+1, n+2,..., \frac{3n-1}{2} \right\} \cup \{ 2n+1, 2n+2,...,3n \}$$

In both case, we observe that all the vertex labels and edge labels are distinct and their union is $\{1, 2, 3,..., 6n+1\}$.

Therefore, f is a super graceful labeling and hence, G is a super graceful graph.

Example 2.5. Super graceful labelings of identifying centre vertex of S_7, DF_6 is given in Figure 5.

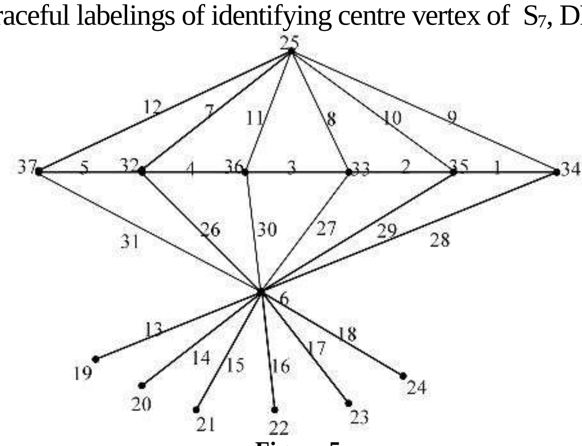

Figure 5

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