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Fixed Point Theorems of Generalized Contraction Mappings in Cone b-metric Spaces

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ABSTRACT

In this paper, we establish and extend some fixed point results for generalized contraction map by using without assumption of normality of cone in complete cone b- metric spaces. Our presented theorems are generalizations of the results J.S. Saluja[3] and P. Kumar *et al*.[4].

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1. INTRODUCTION

Fixed point theory is an extremely dynamic tool in Mathematical analysis. Its plays a basic role in application of many branches of Mathematics. Finding a fixed point of contractive mappings becomes the center of strong research activity. There are many researchers who have worked in fixed point theory of contractive mappings (see for example,[1,2]). In[2] Police Mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle.

In[5], Bakhtin introduced the concept of a b- metric space as a generalization of a metric space and proved the contraction mapping theorem in a b-metric space. He proved the contraction mapping principle in b-metric space that generalized the famous Banach contraction principle in metric spaces.

To overcome the problem of measurable functions with respect to a measure and their convergence, Czerwik[6] needs an extension of metric space. Using this idea, he presented a generalization of renowned Banach fixed point theorem in the b-metric spaces sees for ([7,8,9]). Many authors have studied the extension of fixed point theorem in b-metric spaces.

In 2007, Huang and Zhang[10] generalizing the notion of a metric space by replacing the set of real number by ordered normed spaces, defined a cone metric space and proved some fixed point theorems of contractive mappings defined on these spaces. In 2008, Rezapour and Hamlbarani[11] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric spaces.

Subsequently, several authors have generalized the results of Haung and Zhang[10] and obtained fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric spaces.

Recently, in 2011, Hussein and Shah[12] introduced the concept of cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space. In 2013, Shi and Xu[13], Proved common fixed point theorems for two weakly compatible self-mappings in cone b- metric spaces. In[14], H. Huang and S. Xu. Presented some new examples in cone b-metric spaces and proved some fixed point theorems of contractive mappings without the assumption of normality in cone b-metric spaces. In[15], Reny George and Brian Fisher, obtained a common fixed point theorem of Taskovic type for three mappings in non- normal cone b- metric spaces, which will extend and generalize recent results of Huang and Xu[14], George and Khan[16], Rao *et al*.[17], George *et al*.[18] and also many existing results in metric spaces, b-metric spaces and cone b-metric spaces. Cone b-metric spaces play a useful role in fixed point theory. In 2014, Tiwari, S.K. *et al*.[19], generalized and proved common fixed point theorems for self mapping satisfying a general contractive condition on complete cone b- metric spaces of the results[15]. Kumar, P. *et al*.[20], Expand theorem 1 of the results[14] and proved common fixed point theorems in cone b- metric spaces. In sequel, Tiwari, S.K., *et al*.[21], generalized fixed point theory of cone b- metric spaces.

In fact there exist mappings with common fixed points which are contraction mappings in a cone b-metric space but are not contraction mappings when defined in a cone metric space. The purpose of this paper is to generalize and extend fixed point theorem of generalized contraction mapping in cone b-metric space. Our results extend and improve the results of [3] and [4].

2. PRELIMINARY NOTES

First we recall the definition of cone metric spaces and some properties of theirs[10].

Definition: 2.1 [10]. Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{0\}$;
(ii) $a, b \in R$, $a, b \geq 0$ $x, y \in \boldsymbol{P} \Rightarrow ax+by \in \boldsymbol{P}$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For given a cone $P \subset$ E , we define a Partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if y - $x \epsilon$ $\boldsymbol{P}$. We shall write x $\ll$ y to denote $x \leq y$ but $x \neq y$ to denote y - x ϵ p^0, where p^0 stands for the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ .The least positive number K satisfying the above is called the normal constant of P. The least positive number satisfying the above is called the normal constant P.

In the following, we always suppose that E is a Banach space, P is a cone in E with int $P \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.2[10]: Let X be a non – empty set. Suppose the mapping $d: X \times X \rightarrow E$ satisfies the following condition:

(i) $0 < d(x, y)$ for all $x, y \in$ X and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for $all\ x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 2.3 [10]: Let $E = R^2$, $P = \{(x, y) \in E: x, y \geq 0\}$, $X = R$ and $d: X \times X \rightarrow E$, on defined by $d(x, y) = (|x - y|, \propto |x - y|)$ where $\propto \geq 0$ is a constant. Then (X, d) is a cone metric space.

Example: 2.4. Let E= l^1, P = $\left\{\{x_n\} n \geq 1 \in E: x_n \geq 0, for\ all\ n\right\}$ (X, d) a metric space and $d: X \times X \rightarrow E$, defined by $d(x, y) = \left\{\frac{d(x,y)}{2^n}\right\}$ $n \geq 1$.Then (X, d) is a cone metric space.

Definition 2.5[44]: Let X be a non – empty set. Suppose the mapping $d: X \times X \rightarrow E$ satisfies the following condition:

(i) $\theta < d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]\ for\ all\ x, y, z \in X$.

Then d is called a cone b- metric on X, and the pair (X, d) is called a cone b- metric space.

It is obvious that cone b- metric spaces generalize b-metric spaces and cone metric spaces.

Example 2.5 [45]: Let $E = R^2$, $P = \{(x, y) \in E: x, y \geq 0\}$, $X = R$ and $d: X \times X \rightarrow E$, on defined by $d(x, y) = (|x - y|^p, \propto |x - y|^p)$ where $\propto \geq 0$ and $p > 1$are two constant. Then (X, d) is a cone b- metric space but not a cone metric spaces. In fact, we only need to prove (iii) in Definition 2.5 as follows:

Let $x, y, z \in X$. Set $u = x - z, v = z - y$, so $x - y = u + v$. from the inequality

$(a + b)^p \leq (2\max\{a, b\})^p \leq 2^p(a^p + b^p)$ for all$a, b \geq 0$,

We have

$|x - y|^p = |u + v|^p \leq (|u| + |v|)^p \leq (|u|^p + |v|^p) = 2^p(|x - z|^p + |z - y|^p)$,

This implies that $d(x, y) \leq s[d(x, z) + d(z, y)]$ with s = $2^p > 1$. But

$|x - y|^p = |x - z|^p + |z - y|^p$, is impossible for all $x > z > y$. Indeed, taking account of the inequality

$(a+b)^p > a^p + b^p$ for all $a, b > 0$,we arrive at
$|x-y|^p = |u+v|^p \leq (u+v)^p > u^p + v^p$
$= (x-z)^p + (z-z)^p$
$= |x-z|^p + |z-y|^p$, for all $x > z > y$. thus, (iii) definition 2.5 is not satisfied, $i.e$ (X, d) is not a cone metric space.

Definition 2.6 [44] Let (X, d) be a cone b- metric space, $x \in$ X and $\{x_n\}$ a sequence in X. Then,
(1) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for $all\ n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, $(n \to \infty)$.
(2) $\{x_n\}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is a natural number N such that d $(x_n, x_m) \ll$ c for all $n, m \geq N$.
(3) (X, d) is called a complete cone metric space if every Cauchy sequence in X is Convergent.

Lemma 2.7[12] (i) Let P be a cone and $\{a_n\}$ be a sequence in E. If $c \in intP$ and $\theta \leq a_n \to \theta$ as $(n \to \infty)$, then there exist N such that for all $n > N$, we have $a_n \leq c$.
(ii) Let $x, y, z \in E$, if $x \leq y$ and $y \leq z$ then $x \ll z$.
(iii) Let P be a cone and $a \leq b + c$ for each $c \in intP$, then $a \leq b$.

Lemma 2.8[44] Let P be a cone and $\theta \leq u \leq c$ for each $c \in intP$, then $u = \theta$.

Lemma 2.9[54] Let P be a cone. If $u \in P$ and $u \leq Ku$ for some $0 \leq k \leq 1$then $u = \theta$.

3. MAIN RESULTS

In this section we shall prove some fixed point theorems of generalized contraction mappings in the frame work of cone b-metric spaces.

Theorem 3.1: Let (X, d) be a complete cone b- metric space with the coefficient $s \geq 1$. Suppose $F: X \to X$ be a self mappings satisfying the generalized Contraction map
$d(Fx, Fy) \leq \lambda_1 d(x, y) + \lambda_2 d(x, Fx) + \lambda_3 d(y, Fy) + \lambda_4 [d(x, Fy) + d(y, Fx)]$…..(3.1)
for all $x, y \in X$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0,1)$are constants such that $\lambda_1 + \lambda_2 + \lambda_3 + 2s\lambda_4 < 1$. Then
Then F has a unique fixed point in X. Furthermore, the iterative sequences $\{F^n x\}$ converges to the fixed point.

Proof: Fix $x_0 \in X$ and we construct the iterative sequences $\{x_n\}$, where

$x_n = Fx_{n-1,} n \geq 1$, that is $x_{n+1} = Fx_n = F^{n+1}x$.

We haven
$d(x_{n+1,} x_n) = d(Fx_{n,} Fx_{n-1})$

$\leq \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_n, Fx_n) + \lambda_3 d(x_{n-1}, Fx_{n-1})$
$+\lambda_4[d(d(x_n, Fx_{n-1}) + d(x_{n-1}, Fx_n)]$
$\leq \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x_{n-1}, x_n)$
$+\lambda_4[d(d(x_n, x_n) + d(x_{n-1}, x_{n+1})]$
$= \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x_{n-1}, x_n) + \lambda_4 d(x_{n-1}, x_{n+1})$
$= \lambda_1 d(x_n, x_{n-1}) + \lambda_2 d(x_n, x_{n+1}) + \lambda_3 d(x_{n-1}, x_n)$
$+s\lambda_4[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$
$\leq (\lambda_1 + \lambda_3 + s\lambda_4) d(x_n, x_{n-1}) + (\lambda_2 + s\lambda_4) d(x_n, x_{n+1})$

This implies that

$$d(x_{n+1,} x_n) \leq \frac{(\lambda_1+\lambda_3+s\lambda_4)}{1-(\lambda_2+s\lambda_4)}\, d(x_n, x_{n-1})$$
$$\leq k d(x_n, x_{n-1}), \tag{3.2}$$

where $k = \frac{(\lambda_1+\lambda_3+s\lambda_4)}{1-(\lambda_2+s\lambda_4)} \leq 1$.As $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$.Similarly, we obtain

$$d(x_n, x_{n-1}) \leq k d(x_{n-1}, x_{n-2}) \tag{3.3}$$

Using (3.3) in (3.2), we get

$$d(x_{n+1,} x_n) \leq k^2 d(x_{n-1}, x_{n-2}) \tag{3.4}$$

Continuing this process, we obtain

$$d(x_{n+1,} x_n) \leq k^n d(x_1, x_0) \tag{3.5}$$

For any $m \geq 1, p \geq 1$, we have

$$d(x_m, x_{m+p}) \leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})]$$
$$\leq s d(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3})$$
$$+\ldots\ldots\ldots+s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p})$$
$$\leq s k^m d(x_1, x_0) + s^2 k^{m+1} d(x_1, x_0) + s^3 k^{m+2} d(x_1, x_0)$$
$$+\ldots\ldots\ldots+s^p k^{m+p-1} d(x_1, x_0)$$
$$= s k^m [1 + (sk) + (sk)^2 + (sk)^3 \ldots\ldots\ldots\ldots + (sk)^{p-1}] d(x_1, x_0)$$
$$\leq \frac{sk^m}{1-sk}\, d(x_1, x_0)$$

Let $0 \ll r$ be given. Notice that $\frac{sk^m}{1-sk}\, d(x_1, x_0) \to 0$ as $m \to \infty$ for any p. Making full use of lemma 2.7(i), we find $m_0 \in N$ such that

$$\frac{sk^m}{1-sk}\, d(x_1, x_0) \ll r, \text{ for each } m \geq m_0.$$

Thus,$d(x_m, x_{m+p}) \leq \frac{sk^m}{1-sk}\, d(x_1, x_0) \ll r$, for all $m \geq 1, p > 1$. So, by lemma 2.7(ii) $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b- metric space, there exist $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ Taken $n_0 \in N$ such that $d(x_{n,} x^*) \ll \frac{r(1- s(\lambda_2+\lambda_4))}{s(1+\lambda_1+\lambda_4)}$for all $n \geq n_0$. Hence

$$d(Fx^*, x^*) \leq s[d(Fx^*, Fx_n) + d(Fx_n, x^*)]$$
$$= s d(Fx^*, Fx_n) + s d(Fx_n, x^*)$$
$$= s d(Fx^*, Fx_n) + s d(x_{n+1}, x^*)$$

$$\leq s[\lambda_1 d(x^*, x_n) + \lambda_2 d(x^*, Fx^*) + \lambda_3 sd(x_n, Fx_n)$$
$$+\lambda_4\{d(x^*, Fx_n) + d(x_n, Fx^*)\}] + sd(x_{n+1}, x^*)$$
$$\leq s[\lambda_1 d(x^*, x_n) + \lambda_2 d(x^*, Fx^*) + \lambda_3 d(x_n, x_{n+1})$$
$$+\lambda_4\{d(x^*, x_{n+1}) + d(x_n, Fx^*)\}] + sd(x_{n+1}, x^*)$$
$$\leq s[\lambda_1 d(x^*, x_n) + \lambda_2 d(x^*, Fx^*) + s\lambda_3\{d(x_n, x^*) + d(x^*, x_{n+1})\}$$
$$+\lambda_4\{d(x^*, x_{n+1}) + sd(x_n, x^*) + sd(x^*, Fx^*)\}] + sd(x_{n+1}, x^*)$$
$$= s[\lambda_1 d(x^*, x_n) + \lambda_2 d(x^*, Fx^*) + s\lambda_3\{d(x_n, x^*) + d(x^*, x_{n+1})\}$$
$$+\lambda_4\{d(x^*, x_{n+1}) + sd(x_n, x^*) + sd(x^*, Fx^*)\}] + sd(x_{n+1}, x^*)$$

This implies that

$$d(Fx^*, x^*) \leq s(\lambda_2 + s\lambda_4)d(Fx^*, x^*) + s(\lambda_1 + s\lambda_3 + s\lambda_4)d(x_n, x^*)$$
$$+ s(1 + s\lambda_3 + \lambda_4)\, d(x^*, x_{n+1})$$

1-$s(\lambda_2 + s\lambda_4)d(Fx^*, x^*) \leq s(\lambda_1 + s\lambda_3 + s\lambda_4)d(x_n, x^*) + s(1 + s\lambda_3 + \lambda_4)\ d(x^*, x_{n+1})$

So, $d(Fx^*, x^*) \leq \frac{s(\lambda_1 + s\lambda_3 + s\lambda_4)}{1 - s(\lambda_2 + s\lambda_4)} d(x_n, x^*) + \frac{s(1 + s\lambda_3 + \lambda_4)}{1 - s(\lambda_2 + s\lambda_4)} d(x_n, x_{n+1}) \ll r$ for each $n \geq n_0$.

Then by lemma 2.8 we deduce that $d(Fx^*, x^*) = 0, i.e.,\ Fx^* = x^*$.That is x^* is a fixed point of F.

Now, we show that the fixed point is unique. If $y^*$is another fixed point of F such that $Fy^* = y^*$, then by the given condition (3.1), we have

$$d(x^*, y^*) = d(Fx^*, Fy^*)$$
$$\leq \lambda_1 d(x^*, y^*) + \lambda_2 d(x^*, Fx^*) + \lambda_3 d(y^*, Fy^*)$$
$$+\lambda_4 [d(x^*, Fy^*) + d(y^*, Fx^*)$$
$$\leq (\lambda_1 + 2s\lambda_4)(x^*, y^*),$$ by lemma 2.9, $x^* = y^*$.

Therefore the fixed point of F is unique. This completes the proof of the theorem.

From theorem 3.1, we obtain the following results as corollaries.

Corollary 3.2: Let (X, d) be a complete cone b- metric space with the coefficient $s \geq 1$. Suppose $F: X \to X$ be a self mappings satisfying contractive map

$$d(Fx, Fy) \leq \lambda d(x, y)$$

for all$x, y \in X$, where $\lambda \in [0,1/s]$ is a constant .Then F has a unique fixed point in X.

Proof: The proof of the corollary 3.1 is immediately follows from theorem3.1 by taking $\lambda_2 = \lambda_3 = 0$ and $\lambda_1 = \lambda$. This completes the proof.

Corollary 3.3: Let (X, d) be a complete cone b- metric space with the coefficient $s \geq 1$. Suppose $F: X \to X$ be a self mappings satisfying the generalized Contraction map:

$$d(Fx, Fy) \leq \lambda[d(x, Fx) + d(y, Fy)]$$

for all $x, y \in X$, where $\lambda \in [0,1/2s]$ is a constant . Then F has a unique fixed point in X.

Proof: The proof of the corollary 3.1 is immediately follows from theorem3.1 by taking $\lambda_1 = \lambda_4 = 0$ and $\lambda_2 = \lambda_3 = \lambda$. This completes the proof.

Corollary 3.4: Let (X, d) be a complete cone b- metric space with the coefficient $s \geq 1$. Suppose $F: X \to X$ be a self mappings satisfying the generalized Contraction map:

$$d(Fx, Fy) \leq \lambda[d(x, Fy) + d(y, Fx)]$$

for all $x, y \in X$, where $\lambda \in [0,1/2s]$ is a constant . Then F has a unique fixed point in X.
Proof: The proof of the corollary 3.1 is immediately follows from theorem3.1 by taking $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4 = \lambda$ .This completes the proof.

Remark 3.5 Theorem 3.1 extends the famous Banach contraction principle to that in the setting of cone b- metric spaces.

Theorem 3.6: Let (X, d) be a complete cone b- metric space with the coefficient $s \geq 1$. Suppose $F: X \to X$ be a self mappings satisfying the generalized Contraction map

$$d(Fx, Fy) \leq \lambda_1[d(x, y) + d(x, Fx) + d(y, Fy)] + \lambda_2[d(x, Fy) + d(y, Fx)] \quad (3.6)$$

for all $x, y \in X$, where $\lambda_1, \lambda_2, \in [0,1)$are constants such that $2(\lambda_1 + \lambda_2 s) < 1$. Then
Then F has a unique fixed point in X. Furthermore, the iterative sequences $\{F^n x\}$ converges to the fixed point.

Proof: Fix $x_0 \in X$ and we construct the iterative sequences $\{x_n\}$, where

$$x_n = Fx_{n-1,} n \geq 1, \text{ that is } x_{n+1} = Fx_n = F^{n+1}x .$$

We haven

$$\begin{aligned} d(x_{n+1,} x_n) &= d(Fx_{n,} Fx_{n-1}) \\ &\leq \lambda_1[d(x_n, x_{n-1}) + d(x_n, Fx_n) + d(x_{n-1}, Fx_{n-1})] \\ &+\lambda_2[(d(x_n, Fx_{n-1}) + d(x_{n-1}, Fx_n)] \\ &\leq \lambda_1[d(x_n, x_{n-1}) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &+\lambda_2[d(d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\ &= \lambda_1[d(x_n, x_{n-1}) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \lambda_2 d(x_{n-1}, x_{n+1}) \\ &= \lambda_1[d(x_n, x_{n-1}) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\ &+s\lambda_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &\leq (2\lambda_1 + s\lambda_2)d(x_n, x_{n-1}) + (\lambda_1 + s\lambda_2)d(x_n, x_{n+1}) \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{n+1,} x_n) &\leq \frac{(2\lambda_1 + +s\lambda_2)}{1-(\lambda_1+s\lambda_2)} d(x_n, x_{n-1}) \\ &\leq hd(x_n, x_{n-1}), \end{aligned} \quad (3.7)$$

where $h = \frac{(2\lambda_1 + +s\lambda_2)}{1-(\lambda_1+s\lambda_2)} \leq 1$.As $2(\lambda_1 + \lambda_2 s) < 1$, we obtain that $h < 1$. Similarly, we obtain

$$d(x_n, x_{n-1}) \leq hd(x_{n-1}, x_{n-2}) \quad (3.8)$$

Using (3.8 in (3.7), we get

$$d(x_{n+1,} x_n) \leq h^2 d(x_{n-1}, x_{n-2}) \quad (3.9)$$

Continuing this process, we obtain

$$d(x_{n+1,} x_n) \leq h^n d(x_1, x_0) \quad (3.10)$$

For any $m \geq 1, p \geq 1$, we have

$$d(x_m, x_{m+p}) \leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})]$$

$$\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3})$$
$$+\ldots\ldots\ldots+s^{p-1} d\left(x_{m+p-2}, x_{m+p-1}\right) + s^{p-1} d\left(x_{m+p-1}, x_{m+p}\right)$$
$$\leq sh^m d(x_1, x_0) + s^2 h^{m+1} d(x_1, x_0) + s^3 h^{m+2} d(x_1, x_0)$$
$$+\ldots\ldots\ldots+s^p h^{m+p-1} d(x_1, x_0)$$
$$= sh^m [1 + (sh) + (sh)^2 + (sh)^3 \ldots\ldots\ldots + (sh)^{p-1}] d(x_1, x_0)$$
$$\leq \frac{sh^m}{1-sh} d(x_1, x_0)$$

Let $0 \ll r$ be given. Notice that $\frac{sh^m}{1-sh} d(x_1, x_0) \to 0$ as $m \to \infty$ for any p. Making full use of lemma 2.7(i), we find $m_0 \in N$ such that $\frac{sh^m}{1-sh} d(x_1, x_0) \ll \epsilon$, for each $m \geq m_0$. Thu, $d(x_m, x_{m+p}) \leq$ that $\frac{sh^m}{1-sh} d(x_1, x_0) \ll \epsilon$ for all $m \geq 1, p > 1$. So, by lemma 2.7(ii) $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b- metric space, there exist $u \in X$ such that $x_n \to u$ as $n \to \infty$ Taken $n_0 \in N$ such that $d(x_{n,} u) \ll \frac{r(1- s(\lambda_2 + s\lambda_2)}{s(2\lambda_1 + \lambda_4)}$for all $n \geq n_0$.Hence

$$d(Fu, u) \leq s[d(Fu, Fx_n) + d(Fx_n, u)]$$
$$= sd(Fu, Fx_n), +sd(Fx_n, u)$$
$$\leq s\lambda_1 [d(u, x_n) + d(u, Fu) + d\left(x_{n,} Fx_n\right)]$$
$$+\lambda_2 [d(u, Fx_n) + d(x_n, \mathrm{Fu})] + sd(x_{n+1,} u)$$
$$\leq s[\lambda_1 \{d(u, x_n) + d(u, Fu) + d\left(x_{n,} x_{n+1}\right)\}$$
$$+\lambda_2 s\{d(u, x_{n+1}) + d(x_n, \mathrm{Fu})\}] + sd(x_{n+1,} u)$$
$$= s[\lambda_1 \{d(u, x_n) + d(u, Fu) + d\left(x_{n,} u\right) + d(u, x_{n+1})\}$$
$$+\lambda_2 \{d(u, x_{n+1}) + (sd(x_n, u) + sd(u, Fu))\}] + sd(x_{n+1,} u)$$

This implies that

$$d(Fu, u) \leq s(\lambda_1 + s\lambda_2)\, d(Fu, u) + s(2\lambda_1 + \lambda_2) d(u, x_n)$$
$$+ s(\lambda_1 + \lambda_2 + 1) d(u, x_{n+1})$$

$$1- s(\lambda_2 + s\lambda_2) d(Fu, u) \leq s(2\lambda_1 + \lambda_2) d(u, x_n) + s(\lambda_1 + \lambda_2 + 1) d(u, x_{n+1})$$

So, $d(Fu, u) \leq \frac{s(2\lambda_1 + \lambda_2)}{1- s(\lambda_2 + s\lambda_2)} d(x_n, u) + \frac{s(\lambda_1 + \lambda_2 + 1)}{1- s(\lambda_2 + s\lambda_2)} d(u, x_{n+1}) \ll r$ for each $n \geq n_0$. Then by lemma 2.8 we deduce $d(Fu, u) = 0, i.e.,\ Fu = u$.That is u is a fixed point of F.

Now finally, we show that the fixed point is unique. If $u^*$is another fixed point of F such that $Fu^* = u^*$, then by the given condition (3.6), we have

$$d(u, u^*) = d(Fu, Fu^*)$$
$$\leq \lambda_1 [d(u, u^*) + d(u, Fu) + d(u^*, Fu^*) + \lambda_2 [d(u, Fu^*) + d(u^*, Fu)]$$

$\leq (\lambda_{1+} 2s\lambda_4)(u, u^*)$, by lemma 2.9, $u = u^*$.Therefore the fixed point of F is unique. This completes the proof of the theorem.

Theorem 3.7: Let (X, d) be a complete cone b- metric space with the coefficient $s \geq 1$. Suppose $F: X \to X$ be a self mappings satisfying the Contraction map

$$d(Tx, Ty) \leq \lambda_1 [d(x, Fx) + d(y, Fy)] + \lambda_2 [d(x, Fy) + d(y, Fx)]$$
$$+ \lambda_3 max[d(x, Fx), d(y, Fy), d(x, Fy)] + \lambda_4 [d(x, y) + d(y, Fx)] \qquad (3.11)$$

for all $x, y \in X$, where $\lambda_1, \lambda_{2,} \lambda_3 \in [0,1)$are constants such that $2(\lambda_1 + s\lambda_2 + s\,\lambda_3) + s\,\lambda_4 < 1$. Then F has a unique fixed point in X. Furthermore, the iterative sequences $\{F^n x\}$ converges to the fixed point.

Proof: Fix $x_0 \in X$ and we construct the iterative sequences $\{x_n\}$, where
$x_n = Fx_{n-1,} n \geq 1$, that is $x_{n+1} = Fx_n = F^{n+1}x$.
We haven

$$
\begin{aligned}
d(x_{n+1,}x_n) &= d(Fx_{n,}Fx_{n-1}) \\
&\leq \lambda_1[d(x_n, Fx_n) + d(x_{n-1}, Fx_{n-1})] + \lambda_2[d(x_n, Fx_{n-1}) + d(x_{n-1}, Fx_n)] \\
&\quad +\lambda_3 \max[\, d(x_n, Fx_n), d(x_{n-1}, Fx_{n-1}), d(x_n, Fx_{n-1})] \\
&\quad +\lambda_4[d(x_n, x_{n-1}) +, d(x_{n-1}, Fx_n)] \\
&\leq \lambda_1[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \lambda_2[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\
&\quad +\lambda_3 \max[\, d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n)\,] + \lambda_4[\, d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1})] \\
&= \lambda_1[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \lambda_2 d(x_{n-1}, x_{n+1}) \\
&\quad + \lambda_3 \max[d(x_n, x_{n+1}), d(x_{n-1}, x_n)\,] + \lambda_4[\, d(x_n, x_{n-1}) + d(x_{n-1}, x_{n+1})] \\
&= \lambda_1[d(x_n, x_{n+1}) + d(x_{n-1}, x_n\,)] + s\lambda_2[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&\quad +\lambda_3 \max[\, d(x_n, x_{n+1}), d(x_{n-1}, x_n)] \\
&\quad +\lambda_4[\, d(x_n, x_{n-1}) + s\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}]
\end{aligned}
$$

Therefore,

$$
1\text{-}[\lambda_1 + s\lambda_2 + \lambda_3 + s\lambda_4]\, d(x_{n+1,}x_n) \leq [\lambda_1 + s\lambda_2 + \lambda_3 + (1+s)\lambda_4] d(x_{n-1}, x_n\,)
$$

$$
\leq (2\lambda_1 + s\lambda_2) d(x_n, x_{n-1}) + (\lambda_1 + s\lambda_2 + \lambda_3 + s\lambda_4) d(x_n, x_{n+1})
$$

Hence,

$$
\begin{aligned}
d(x_{n+1,}x_n) &\leq \frac{[\lambda_1 + s\lambda_2 + \lambda_3 + (1+s)\lambda_4]}{[\lambda_1 + s\lambda_2 + \lambda_3 + s\lambda_4]}\, d(x_n, x_{n-1}) \\
&\leq h d(x_n, x_{n-1}), \qquad (3.12)
\end{aligned}
$$

where $h = \frac{[\lambda_1 + s\lambda_2 + \lambda_3 + (1+s)\lambda_4]}{[\lambda_1 + s\lambda_2 + \lambda_3 + s\lambda_4]} \leq 1$.As $2(\lambda_1 + s\lambda_2 + s\,\lambda_3) + s\lambda_4 < 1$we obtain that $h < 1$.
Similarly, we obtain

$$
d(x_n, x_{n-1}) \leq h d(x_{n-1}, x_{n-2}) \qquad (3.13)
$$

Using (3.13) in (3.12), we get

$$
d(x_{n+1,}x_n) \leq h^2 d(x_{n-1}, x_{n-2}) \qquad (3.14)
$$

Continuing this process, we obtain

$$
d(x_{n+1,}x_n) \leq h^n d(x_1, x_0) \qquad (3.15)
$$

For any $m \geq 1, p \geq 1$, we have

$$
\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&\leq s d(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) \\
&\quad + \ldots\ldots\ldots + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq s h^m d(x_1, x_0) + s^2 h^{m+1} d(x_1, x_0) + s^3 h^{m+2} d(x_1, x_0) \\
&\quad + \ldots\ldots\ldots + s^p h^{m+p-1} d(x_1, x_0) \\
&= s h^m [1 + (sh) + (sh)^2 + (sh)^3. \ldots \ldots \ldots . + (sh)^{p-1}] d(x_1, x_0)
\end{aligned}
$$

$$\leq \frac{sh^m}{1-sh}\, d(x_1, x_0)$$

Let $0 \ll \epsilon$ be given. Notice that $\frac{sh^m}{1-sh}\, d(x_1, x_0) \to 0$ as $m \to \infty$ for any p. Making full use of lemma 2.7(i), we find $m_0 \in N$ such that $\frac{sh^m}{1-sh}\, d(x_1, x_0) \ll \epsilon$, for each $m \geq m_0$. Thu, $d(x_m, x_{m+p}) \leq$ that $\frac{sh^m}{1-sh}\, d(x_1, x_0) \ll \epsilon$ for all $m \geq 1, p > 1$. So, by lemma 2.7(ii) $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete cone b- metric space, there exist $v \in X$ such that $x_n \to v$ as $n \to \infty$ Taken $n_0 \in N$ such that $d(x_{n,} v) \ll \frac{\epsilon\ [1 - s\lambda_1 + s^2\lambda_2 + s^2\lambda_4)]}{s^2\ (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}$ for all $n \geq n_0$. Hence

$$\begin{aligned}
d(Fv, v) &\leq s[d(Fv, Fx_n) + d(Fx_n, v)] \\
&= sd(Fv, Fx_n), + sd(Fx_n, v) \\
&\leq s[\lambda_1\{d(v, Fv) + d(x_{n,} Fx_n)\} + \lambda_2\{d(v, Fx_n) + d(x_n, \mathrm{F}v)\} \\
&\quad + \lambda_3 max\{d(v, Fv), d(x_n, \mathrm{F}x_n),\ d(v, Fx_n)\} + \lambda_4\{d(v, x_n) + d(x_n, \mathrm{F}v)\}] \\
&\quad + sd(x_{n+1,} v) \\
&\leq s[\lambda_1\{d(v, Fv) + d(x_{n,} x_{n+1})\} + \lambda_2\{d(v, x_{n+1}) + d(x_n, \mathrm{F}v)\} \\
&\quad + \lambda_3 max\{d(v, Fv), d(x_{n,} x_{n+1}), d(v, x_{n+1})\} + \lambda_4\{d(v, x_n) + d(x_n, \mathrm{F}v)\}] \\
&\quad + sd(x_{n+1,} v) \\
&\leq (\lambda_1 + s^2\lambda_2 + s^2\lambda_4)\, d(Fv, v) + s^2\, (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\, d(x_n, v) \\
&\quad + \{s(1 + \lambda_2) + s^2(\lambda_1 + \lambda_3)\}\, d(v, x_{n+1})
\end{aligned}$$

Therefore,

$$\begin{aligned}
[1 - s\lambda_1 + s^2\lambda_2 + s^2\lambda_4)]\, d(Fv, u) \leq{}& s^2\, (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\, d(x_n, v) \\
&+ \{s(1 + \lambda_2) + s^2(\lambda_1 + \lambda_3)\}\, d(v, x_{n+1})
\end{aligned}$$

So, $d(Fv, v) \leq \frac{s^2\ (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}{[1 - s\lambda_1 + s^2\lambda_2 + s^2\lambda_4)]}\, d(x_n, u) + \frac{\{s(1+\lambda_2) + s^2(\lambda_1 + \lambda_3)\}}{[1 - s\lambda_1 + s^2\lambda_2 + s^2\lambda_4)]}\, d(v, x_{n+1}) \ll \epsilon$ for each $n \geq n_0$. Then by lemma 2.8 we deduce $d(Fv, v) = 0, i.e.,\ Fv = v$.That is v is a fixed point of F.

Now finally, we show that the fixed point is unique. If $v^*$is another fixed point of F such that $Fv^* = v^*$, then by the given condition (3.11), we have

$$\begin{aligned}
d(v, v^*) &= d(Fv, Fv^*) \\
&\leq \lambda_1[d(v, Fv) + d(v^*, Fv^*)] + \lambda_2[d(v, Fv^*) + d(v^*, \mathrm{F}v)\} \\
&\quad + \lambda_3 max\{d(v, Fv), d(v^*, Fv^*), d(v, Fv^*)\} + \lambda_4\, [d(v, v^*) + d(v^*, \mathrm{F}v)] \\
&\leq [2s(\lambda_2 + \lambda_4) + \lambda_3]\, d(v, v^*) \\
&\leq 2(\lambda_1 + s\lambda_2 + s\,\lambda_3) + s\,\lambda_4]\, d(v, v^*)
\end{aligned}$$

Owing to $0 \leq [2(\lambda_1 + s\lambda_2 + s\,\lambda_3) + s\,\lambda_4] < 1$. Then by lemma 2.9, $v = v^*$ Therefore the fixed point of F is unique. This completes the proof of the theorem.

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