Journal of Computer and Mathematical Sciences, *Vol.10(2),310-315 February 2019* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Common Fixed Point Theorem using E.A. Property

Ravi S[1] and V.Srinivas[2]

[1]Department of Mathematics,
University Post Graduate College, Secunderabad,
Osmania University, Hyderabad-500003 (Telangana), INDIA.
[2]Department of Mathematics,
University College of Science, Saifabad,
Osmania University, Hyderabad-50009 (Telangana), INDIA.
(Corresponding author: Ravi S)
email: [1] ravisriramula@gmail.com, [2]srinivasmaths4141@gmail.com.

(Received on: February 7, 2019)

ABSTRACT

For this paper we prove a common fixed point theorem in a metric space which is a generalization of Bijendra Singh and M.S. Chauhan using some weaker conditions namely E.A. Property and coincidence points.

AMS (2010) Mathematics Classification: 54H25, 47H10.

Keywords: Fixed point, self maps, coincidence point, E.A.Property.

1. INTRODUCTION

Fixed point theory is an important area of functional analysis. The study of common fixed point of mappings satisfying contractive type condition has been a very active field of research. G.Jungck[1] introduced the concept of compatible maps which is weaker than weakly commuting maps. After wards Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible maps. This concept has been frequently used to prove existence theorem in common fixed point theory.

In 2002, Aamri and Moutawakil[10] generalized the notion of noncompatible mapping to the E.A. property. It was pointed out in[10] that the property E.A. is independent of continuity requirements besides minimizing the commutativity conditions of the maps to the commutativity at their points of coincidence. Moreover, the E.A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range.

In this paper we prove a common fixed point theorem for four self maps in which two pairs are satisfying the Property E.A. and have coincidence points.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. A and S are two self maps of a metric space (X,d) are said to be *compatible mappings* if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.2. A and S are two self maps on metric space X, if $Au = Su = w, w \in X$ for some u in X then u is called a coincidence point and w is called point of coincidence of the pair (A,S). For example

Example 2.3. Let $X = [0,\infty)$ with usual metric, define self maps A and S on X as $A(x) = \frac{1+x}{3}$ and $S(x) = \frac{1+2x^2}{3}$, then

$$A(0) = S(0) = \frac{1}{3} \text{ and } A\left(\frac{1}{2}\right) = \frac{1+\frac{1}{2}}{3} = \frac{1}{2}. \text{ Also } S\left(\frac{1}{2}\right) = \frac{1+2\frac{1}{4}}{3} = \frac{\frac{3}{2}}{3} = \frac{1}{2}.$$

Therefore 0 and $\frac{1}{2}$ are coincidence points.

Definition 2.4. A and S are two self maps of metric space (X,d) are said to satisfy the E.A. Property if there exists a sequence $< x_n >$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. For example

Example 2.5. Let (X,d) be a metric space where $X = [0,2]$ and d(x,y)=|x-y|. We define self maps A and S as

$$A(x) = \begin{cases} 0 & if\ 0 \le x < \frac{1}{4} \\ x & if\ \frac{1}{4} \le x \le 2 \end{cases} \quad \text{and} \quad S(x) = \begin{cases} \frac{1}{4} & if\ 0 \le x < \frac{1}{4} \\ x & if\ \frac{1}{4} \le x \le 2 \end{cases}$$

Define the sequence $\{x_n\}$ as $x_n = \frac{1}{4} + \frac{1}{n}$, for $n \ge 1$, then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(\frac{1}{4}+\frac{1}{n}\right) = \frac{1}{4} \text{ and } \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} S\left(\frac{1}{4}+\frac{1}{n}\right) = \frac{1}{4}.$$

Therefore $\lim_{n\to\infty} Ax_n = \frac{1}{4} = \lim_{n\to\infty} Sx_n$.

Hence the pair (A,S) satisfy the Property E.A.

Bijenrda Singh and M.S.Chauhan[5] proved the following theorem.

Theorem 2.6. Suppose A, B, S and T are self mappings from a complete metric space (X,d) into itself satisfying the following conditions

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ (2.6.1)

one of A,B,S and T is continuous (2.6.2)

$[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty) + d(By,Sx)d(Ax,Ty)]$ (2.6.3)

$+k_2[d(Ax,Sx)d(Ax,Ty) + d(By,Ty)d(By,Sx)]$

where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

the pairs (A,S) and (B,T) are compatible on X. (2.6.4)

Further, if X is a complete metric space then A,B,S and T have a unique common fixed point in X.

3. MAIN RESULT

Theorem 3.1. Suppose A, B, S and T are self maps from a metric space (X,d) into itself satisfying the following conditions

$$[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty) + d(By,Sx)d(Ax,Ty)] + k_2[d(Ax,Sx)d(Ax,Ty) + d(By,Ty)d(By,Sx)] \quad (3.1.1)$$

for all x,y in X where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

the pairs (A,S) and (B,T) satisfy the Property E.A. (3.1.2)

the pairs (A,S) and (B,T) have a coincidence point. (3.1.3)

Then A,B,S and T have a unique common fixed point in X. (3.1.4)

Proof:

Since the pairs (A,S) and (B,T) are satisfy the property E.A., then there exists two sequences $\{x_{2n}\}$ and $\{y_{2n}\}$ in X such that $\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Sx_{2n} = u$, $u \in X$ (3.1.5)

and $\lim_{n\to\infty} By_{2n} = \lim_{n\to\infty} Ty_{2n} = v$, $v \in X$ (3.1.6)

Therefore $Au = Su$ and $Bv = Tv$ that is u is a coincidence point of A and S and v is a coincidence point of B and T.

Now we have to show that $u = v$.

Putting $x = x_{2n}, y = y_{2n}$ in the condition (3.1.1), we have

$$[d(Ax_{2n}, By_{2n})]^2 \le k_1[d(Ax_{2n}, Sx_{2n})d(By_{2n}, Ty_{2n}) + d(By_{2n}, Sx_{2n})d(Ax_{2n}, Ty_{2n})] + k_2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Ty_{2n}) + d(By_{2n}, Ty_{2n})d(By_{2n}, Sx_{2n})]$$

using the conditions $Ax_{2n}, Sx_{2n} \to u$ and $By_{2n}, Ty_{2n} \to v$ as $n \to \infty$, then we get

$$[d(u,v)]^2 \le k_1[d(u,u)d(v,v) + d(v,u)d(u,v)] + k_2[d(u,u)d(u,v) + d(v,v)d(v,u)]$$

and this gives

$$[d(u,v)]^2 \le k_1[d(u,v)]^2$$

this implies

$(1-k_1)[d(u,v)]^2 \le 0$, since $0 \le k_1 + 2k_2 < 1$

$d(u,v) = 0$ implies that $u = v$.

Now we claim that $Au = u$, put $x = u, y = y_{2n}$ in the condition (3.1.1), we have

$$[d(Au, By_{2n})]^2 \le k_1[d(Au,Su)d(By_{2n},Ty_{2n}) + d(By_{2n},Su)d(Au,Ty_{2n})] + k_2[d(Au,Su)d(Au,Ty_{2n}) + d(By_{2n},Ty_{2n})d(By_{2n},Su)]$$

letting $n \to \infty$ and using the conditions $By_{2n}, Ty_{2n} \to v$ and $Au = Su$, we get

$$[d(Au,v)]^2 \le k_1[d(Au,Au)d(v,v) + d(v,Au)d(Au,v)] + k_2[d(Au,Au)d(Au,v) + d(v,v)d(v,Au)]$$

this gives

$$[d(Au,v)]^2 \le k_1[d(Au,v)]^2$$

and this implies

$(1-k_1)[d(Au,v)]^2 \le 0$, since $0 \le k_1 + 2k_2 < 1$

$d(Au,v) = 0$ implies $Au = v$.

Therefore $Au = u = v$.

Since $Bv = Tv$ gives that $Bu = Tu$ by using $u = v$.

Again we claim that $Bu = u$, put $x = u, y = u$ in the condition (3.1.1), we have

$$[d(Au, Bu)]^2 \le k_1[d(Au,Su)d(Bu,Tu) + d(Bu,Su)d(Au,Tu)] + k_2[d(Au,Su)d(Au,Tu) + d(Bu,Tu)d(Bu,Su)]$$

using the conditions $Au = Su = u$ and $Bu = Tu$, we get

$$[d(u,Bu)]^2 \le k_1[d(u,u)d(u,Bu) + d(Bu,u)d(u,Bu)] + k_2[d(u,u)d(u,Tu) + d(Bu,Bu)d(Bu,u)]$$

and this gives

$$[d(u,Bu)]^2 \le k_1[d(Bu,u)]^2$$

this implies

$(1-k_1)[d(u,Bu)]^2 \leq 0$, since $0 \leq k_1 + 2k_2 < 1$

$d(Bu,u) = 0$ implies $Bu = u$.

Hence $Bu = Tu = u$.

Since $Au = Bu = Su = Tu = u$, we get u is a common fixed point of A, B, S and T. The uniqueness of the fixed point can be easily proved.

The above Theorem can be validating by using following example.

Example 3.2. Let $X = [0,1]$ with $d(x,y) = |x-y|$. Define self maps of A, B, S and T of X by

$$A(x) = B(x) = \begin{cases} \frac{1}{2} & if \;\; 0 \leq x \leq \frac{1}{2} \\ \frac{3}{4} & if \;\; \frac{1}{2} < x \leq 1 \end{cases} \quad \text{and} \quad S(x) = T(x) = \begin{cases} 1-x & if \;\; 0 \leq x \leq \frac{1}{2} \\ \frac{4}{5} & if \;\; \frac{1}{2} < x \leq 1 \end{cases}$$

Clearly $A\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = \frac{1}{2}$ so that $AS\left(\frac{1}{2}\right) = A\left[S\left(\frac{1}{2}\right)\right] = A\left[\frac{1}{2}\right] = \frac{1}{2}$ and

$SA\left(\frac{1}{2}\right) = S\left[A\left(\frac{1}{2}\right)\right] = S\left[\frac{1}{2}\right] = \frac{1}{2}$.

Consequently the pair (A,S) is commuting at the coincidence point $\frac{1}{2}$.

Define a sequence $\{x_n\}$ as $x_n = \frac{1}{2} - \frac{1}{3n}$ for $n \geq 1$, then

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(\frac{1}{2} - \frac{1}{3n}\right) = \frac{1}{2}$ and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} S\left(\frac{1}{2} - \frac{1}{3n}\right) = \lim_{n\to\infty}\left(1 - \left(\frac{1}{2} - \frac{1}{3n}\right)\right) = \frac{1}{2}$

Therefore $\lim_{n\to\infty} Ax_n = \frac{1}{2} = \lim_{n\to\infty} Sx_n$ which shows that the pair (A,S) satisfy the property E.A.

Remark 3.3. From the example given above, clearly the pairs (A, S) and (B,T) are satisfy the property E.A. and have coincidence point $\frac{1}{2}$. Further the condition (3.1.1) holds for the values of $k_1, k_2 \geq 0$, satisfying the condition $0 \leq k_1 + 2k_2 < 1$. It can be seen that X is not a complete metric space and $\frac{1}{2}$ is a common fixed point of A, B, S and T. We observe that '$\frac{1}{2}$' is the unique common fixed point of A, B, S and T.

REFERENCES

1. G.Jungck, Compatible mappings and fixed points, *Internat. J. Math. & Math. Sci.*, 9, 771-778 (1986).
2. R.P.Pant, A common fixed point theorem under a new condition, *Indian J. of Pure and App. Math.*, 30(2), 147-152 (1999).
3. G.Jungck, Compatible mappings and common fixed points, *Internat. J. Math. & Math. Sci.*, 11, 285-288 (1988).
4. G.Jungck and B.E. Rhoades, Fixed point for set valued functions without continuity, *Indian J.Pure. Appl.Math.*, 29(3), 227-238 (1998).
5. B.Singh and S.Chauhan, On common fixed points of four mappings, *Bull. Cal. Math. Soc.*, 88, 301-308 (1998).
6. R.P.Pant,R.K.Bisht and D.Arora, Weak reciprocal continuity and fixed point theorems, *Ann. Univ. Ferrara Sez. VIII Sci. Mat*.57, No.1, 181-190 (2011).
7. V.Srinivas and Ravi Sriramula, Formation of a common fixed point theorem using compatible mappings of type (E), *Arya Bhatta Journal of Mathematics & Informatics*, 9(1), 15-20 (2017).
8. Singh,S.L. and Mishra,S.N., Coincidences and fixed points of reciprocally continuous and compatible hybrid maps, *Internat. J. Math. & Math. Sci*. 30(10), 627-635, (2002).
9. R.P.Pant, Common fixed points of four mappings, *Bull. Calcutta Math. Soc*. 90, 281-286 (1998).
10. M. Aamri and D. El Moutawakil, “Some new common fixed point theorems under strict contractive conditions,” *Journal of Mathematical Analysis and Applications*, vol. 270, no. 1, pp. 181–188, (2002).