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On the Growth Properties of Generalized Iterated Entire Functions

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ABSTRACT

We study the generalized iteration of entire functions and investigate the growth properties of iterated entire functions of finite iterated order. Here we prove some results on the growth of iterated entire functions of finite iterated order. The results improve and generalize some earlier results.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS

In order to study the growth properties of generalized iterated entire functions, it is very much necessary to mention some relevant notations and definitions. For standard notations and definitions we refer to[3].

Notation 1.1[10] Let $log^{[0]}r = r$, $exp^{[0]}r = r$ and for positive integer p, $log^{[P]}r = log(log^{[p-1]}r)$, $exp^{[p]}r = exp(exp^{[p-1]}r)$.

Definition 1.2 The order $\rho_{[f]}$ and lower order $\lambda_{[f]}$ of a meromorphic function f is defined as

$$\rho_{[f]} = \lim_{r\to\infty} sup \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_{[f]} = \lim_{r\to\infty} inf \ \frac{\log T(r,f)}{\log r}.$$

If f is an entire function, then one can easily verify that

$$\rho_{[f]} = \lim_{r\to\infty} sup \ \frac{\log^{[2]} M(r,f)}{\log r}$$

and

$$\lambda_{[f]} = \lim_{r\to\infty} inf \ \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 1.3 A function $\lambda_{[f]}(r)$ is called a lower proximate order of a meromorphic function f if

(i) $\lambda_{[f]}(r)$ is nonnegative and continuous for $r \geq r_0$, say;

(ii) is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda'_{[f]}(r-0)$ and $\lambda'_{[f]}(r+0)$ exist;

(iii) $\lim_{r\to\infty} \lambda_{[f]}(r) = \lambda_{[f]} < \infty$;

(iv) $\lim_{r\to\infty} r\lambda'_{[f]}(r) \log r = 0$

and

(v) $\lim_{r\to\infty} inf \ \frac{T(r,f)}{r^{\lambda_{[f]}(r)}} = 1$.

Definition 1.4 [1] Let f and g be two entire functions defined in the open complex plane and $\xi \in (0,1]$. Then the generalized iterations of f with respect to g are defined as follows:

$f_{[1,g]}(z) = (1-\xi)z + \xi f(z)$
$f_{[2,g]}(z) = (1-\xi)g_{[1,f]}(z) + \xi f(g_{[1,f]}(z))$
$f_{[3,g]}(z) = (1-\xi)g_{[2,f]}(z) + \xi f(g_{[2,f]}(z))$
… … … … … … … … … … … … … … … … … … ….
… … … … … … … … … … … … … … … … … … ….
$f_{[n,g]}(z) = (1-\xi)g_{[n-1,f]}(z) + \xi f(g_{[n-1,f]}(z))$

and so are

$g_{[1,f]}(z) = (1-\xi)z + \xi g(z)$
$g_{[2,f]}(z) = (1-\xi)f_{[1,g]}(z) + \xi g(f_{[1,g]}(z))$
$g_{[3,f]}(z) = (1-\xi)f_{[2,g]}(z) + \xi g(f_{[2,g]}(z))$
… … … … … … … … … … … … … … … … … … ….
… … … … … … … … … … … … … … … … … … ….
$g_{[n,f]}(z) = (1-\xi)f_{[n-1,g]}(z) + \xi g(f_{[n-1,g]}(z))$

Clearly all $f_{[n,g]}(z)$ and $g_{[n,f]}(z)$ are entire functions.

For two non-constant entire functions $f(z)$ and $g(z)$, it is well known that

$$\log M(r, f(g)) \leq \log M(M(r,g), f). \quad (1)$$

Let $f(z)$ and $g(z)$ be any two transcendental entire functions defined in the open complex plane $\mathbb{C}$. In 1970, Clunie[2] proved that $\lim_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{\log T(r, f\circ g)}{T(r,g)} = 0$. In 1985, Singh[11] proved some comparative growth properties of $\log T(r, f\circ g)$ and $T(r,f)$. Afterwards Lahiri[5] proved some results on the comparative growth of $\log T(r, f\circ g)$ and $T(r,g)$.

Recently Lahiri and Datta[6] made a close investigation on the comparative growth properties of $\log T(r, f\circ g)$ and $T(r,g)$. They[6] also proved some results on the comparative growth properties of $\log\log T(r, f\circ g)$ and $T(r, f^{(k)})$.

In this paper, we study the growth of generalized iterated entire functions and prove some results which generalize and improve some earlier results.

2. LEMMAS

The following lemmas will be needed in the sequel.

Lemma 2.1[3] Let $f(z)$ be an entire function. For $0 \leq r < R < \infty$, we have
$$T(r,f) \leq \log^{+} M(r,f) \leq \frac{R+r}{R-r} T(R,f).$$

Lemma 2.2 [3] Let $f(z)$ and $g(z)$ be two transcendental entire functions. Then
$\frac{T(r,f)}{T(r,g(f))} \to 0$ as $r \to \infty$.

Lemma 2.3[8] Let $f(z)$ and $g(z)$ be two entire functions. If $M(r,g) > \frac{(2+\sigma)}{\sigma}|g(0)|$ for any $\sigma > 0$, then
$$T(r, f(g)) < (1+\sigma) T(M(r,g), f).$$
In particular if $g(0) = 0$, then $T(r, f(g)) \leq T(M(r,g), f)$ for all $r > 0$.

Lemma 2.4[9] Let $f(z)$ and $g(z)$ be two entire functions. Then we have
$$T(r, f(g)) \geq \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Lemma 2.5[4] Let $f(z)$ be an entire function. Then for $k > 2$,
$$\liminf_{r\to\infty} \frac{\log^{[k-1]} M(r,f)}{\log^{[k-2]} T(r,f)} = 1.$$

Lemma 2.6[6] Let $f(z)$ be an meromorphic function. Then for $\delta(>0)$, the function $r^{\lambda_{[f]}+\delta-\lambda_{[f]}(r)}$ is an increasing function of r.

Lemma 2.7[7] Let f be an entire function of finite lower order. If there exist entire functions a_i $(i = 1,2,3\ldots\ldots\ldots\ldots n; n \leq \infty)$ satisfying $T(r, a_i) = o(T(r,f))$ and $\sum_{i=1}^{n} \delta(a_i, f) = 1$, then

$$\lim_{r\to\infty}\frac{T(r,f)}{\log M(r,f)}=\frac{1}{\pi}.$$

Lemma 2.8[9] Let $f(z)$ and $g(z)$ be two non-constant entire functions such that $0<\lambda_{[f]}\leq\rho_{[f]}<\infty$ and $0<\lambda_{[g]}\leq\rho_{[g]}<\infty$. Then for any σ $(0<\sigma<\min\{\lambda_{[f]},\lambda_{[g]}\})$

$$\log^{[n-1]}T(r,f_{[n,g]})\leq\begin{cases}(\rho_{[f]}+\sigma)(1+O(1))\log M(r,g)+O(1), & when\ n\ is\ even\\(\rho_{[g]}+\sigma)(1+O(1))\log M(r,f)+O(1), & when\ n\ is\ odd\end{cases}$$

and

$$\log^{[n-1]}T(r,f_{[n,g]})\geq\begin{cases}(\lambda_{[f]}-\sigma)(1+O(1))\log M\left(\frac{r}{4^{n-1}},g\right)+O(1), & when\ n\ is\ even\\(\lambda_{[g]}-\sigma)(1+O(1))\log M\left(\frac{r}{4^{n-1}},f\right)+O(1), & when\ n\ is\ odd,\end{cases}$$

for all large values of r.

Proof. We get from Lemma 2.2, Lemma 2.3 and (1) for $\sigma(>0)$ and for all large values of r,

$$T(r,f_{[n,g]})\leq T(r,g_{[n-1,f]})+T\left(r,f(g_{[n-1,f]})\right)+O(1)$$
$$\leq(1+0(1))T\left(r,f(g_{[n-1,f]})\right)$$
$$\leq 2T(M(r,g_{[n-1,f]}),f)$$

Therefore using Definition 1.1 we have

$$\log T(r,f_{[n,g]})\leq\log T(M(r,g_{[n-1,f]}),f)+O(1)$$
$$\leq(\rho_{[f]}+\sigma)\log M(r,g_{[n-1,f]})+O(1)$$

Hence

$$\log^{[2]}T(r,f_{[n,g]})\leq\log^{[2]}M(r,g_{[n-1,f]})+O(1)$$
$$\leq\log\{\log M(r,f_{[n-2,g]})+\log M(r,g(f_{[n-2,g]}))+O(1)\}+O(1)$$
$$\leq\log\{\log M(M(r,f_{[n-2,g]}),g)+\log M(M(r,f_{[n-2,g]}),g)+O(1)\}+O(1)$$
$$\leq\log\log M(M(r,f_{[n-2,g]}),g)+O(1)$$
$$\leq(\rho_{[g]}+\sigma)\log M(r,f_{[n-2,g]})+O(1)$$

… …

… …

Therefore

$$\log^{[n-1]}T(r,f_{[n,g]})\leq(\rho_{[f]}+\sigma)\log M(r,g_{[1,f]})+O(1)$$
$$\leq(\rho_{[f]}+\sigma)\{\log M(r,z)+\log M(r,g)+O(1)\}+O(1)$$
$$\leq(\rho_{[f]}+\sigma)(1+O(1))\log M(r,g)+O(1), when\ n\ is\ even$$

Similarly,

$$\log^{[n-1]}T(r,f_{[n,g]})\leq(\rho_{[g]}+\sigma)(1+O(1))\log M(r,f)+O(1), when\ n\ is\ odd$$

Again for σ ($0 < \sigma < \min\{\lambda_{[f]}, \lambda_{[g]}\}$), we get from Lemma 2.2 and Lemma 2.4 for all large values of r

$$\begin{aligned} T(r, f_{[n,g]}) &\geq T(r, f(g_{[n-1,f]})) - T(r, g_{[n-1,f]}) + O(1) \\ &= (1+0(1)) T\left(r, f(g_{[n-1,f]})\right) \\ &\geq (1+0(1)) \frac{1}{3} \log M(\frac{1}{8} M(\frac{r}{4}, g_{[n-1,f]}) + O(1), f) \\ &\geq (1+0(1)) \frac{1}{3} \left[\frac{1}{8} M\left(\frac{r}{4}, g_{[n-1,f]}\right) + O(1)\right]^{\lambda_{[f]}-\sigma} \\ &\geq (1+0(1)) \frac{1}{3} \left[\frac{1}{9} M\left(\frac{r}{4}, g_{[n-1,f]}\right)\right]^{\lambda_{[f]}-\sigma} \qquad (2) \end{aligned}$$

Hence

$$\begin{aligned} \log T(r, f_{[n,g]}) &\geq (\lambda_{[f]} - \sigma) \log M\left(\frac{r}{4}, g_{[n-1,f]}\right) + O(1) \\ &\geq (\lambda_{[f]} - \sigma) T\left(\frac{r}{4}, g_{[n-1,f]}\right) + O(1) \\ &\geq (\lambda_{[f]} - \sigma)[T\left(\frac{r}{4}, g(f_{[n-2,g]})\right) - T\left(\frac{r}{4}, f_{[n-2,g]}\right) + O(1)] + O(1) \\ &\geq (\lambda_{[f]} - \sigma)(1+0(1)) T\left(\frac{r}{4}, g(f_{[n-2,g]})\right) + O(1) \\ &\geq (\lambda_{[f]} - \sigma) \frac{1}{3} \log M(\frac{1}{8} M(\frac{r}{4^2}, f_{[n-2,g]}) + O(1), g) + O(1) \\ &\geq (\lambda_{[f]} - \sigma) \frac{1}{3} [\frac{1}{8} M(\frac{r}{4^2}, f_{[n-2,g]}) + O(1)]^{\lambda_{[g]}-\sigma} + O(1) \\ &\geq (\lambda_{[f]} - \sigma) \frac{1}{3} [\frac{1}{9} M(\frac{r}{4^2}, f_{[n-2,g]})]^{\lambda_{[g]}-\sigma} + O(1) \end{aligned}$$

So

$$\log^{[2]} T(r, f_{[n,g]}) \geq (\lambda_{[g]} - \sigma) \log M\left(\frac{r}{4^2}, f_{[n-2,g]}\right) + O(1)$$

… ..

… ..

Therefore

$$\log^{[n-2]} T(r, f_{[n,g]}) \geq (\lambda_{[g]} - \sigma) \log M\left(\frac{r}{4^{n-2}}, f_{[2,g]}\right) + O(1) \qquad (3)$$

$$\begin{aligned} \log^{[n-1]} T(r, f_{[n,g]}) &\geq (\lambda_{[f]} - \sigma) \log M\left(\frac{r}{4^{n-1}}, g_{[1,f]}\right) + O(1) \\ &\geq (1+0(1))(\lambda_{[f]} - \sigma) \log M\left(\frac{r}{4^{n-1}}, g\right) + O(1), when\ n\ is\ even. \end{aligned}$$

Similarly

$$\log^{[n-1]} T(r, f_{[n,g]}) \geq (1+0(1))(\lambda_{[g]} - \sigma) \log M\left(\frac{r}{4^{n-1}}, f\right) + O(1), when\ n\ is\ odd.$$

This proves the lemma.

3. THEOREMS

Theorem 3.1 Let f and g be two non-constant entire functions having finite lower orders. Then

(i) $\liminf_{r\to\infty}\frac{\log^{[n-1]} T(r,f_{[n,g]})}{T(r,g)} \le 3\rho_{[f]}2^{\lambda_{[g]}},\ when\ n\ is\ even,$

(ii) $\limsup_{r\to\infty}\frac{\log^{[n-1]} T(r,f_{[n,g]})}{T(r,g)} \ge \frac{\lambda_{[f]}}{(4^{n-1})^{\lambda_{[g]}}},when\ n\ is\ even,$

(iii) $\liminf_{r\to\infty}\frac{\log^{[n-1]} T(r,f_{[n,g]})}{T(r,f)} \le 3\rho_{[g]}2^{\lambda_{[f]}},\ when\ n\ is\ odd$

and

(iv) $\limsup_{r\to\infty}\frac{\log^{[n-1]} T(r,f_{[n,g]})}{T(r,f)} \ge \frac{\lambda_{[g]}}{(4^{n-1})^{\lambda_{[f]}}},when\ n\ is\ odd.$

Proof. We may clearly assume $0 < \lambda_{[f]} \le \rho_{[f]} < \infty$ and $0 < \lambda_{[g]} \le \rho_{[g]} < \infty$. Now from Lemma 2.8, for arbitrary $\sigma > 0$,

$$\log^{[n-1]} T(r,f_{[n,g]}) \le (1+O(1))(\rho_{[f]}+\sigma)\log M(r,g) + O(1), \tag{4}$$

when n is even.

Let $0 < \sigma < \min\{1,\ \lambda_{[f]},\ \lambda_{[g]}\}$. Since

$$\lim_{r\to\infty} inf\ \frac{T(r,g)}{r^{\lambda_{[g]}(r)}} = 1,$$

there exists a sequence of values of r tending to infinity for which

$$T(r,g) < (1+\sigma)r^{\lambda_{[g]}(r)} \tag{5}$$

and for all large value of r

$$T(r,g) > (1-\sigma)r^{\lambda_{[g]}(r)} \tag{6}$$

Thus for a sequence of values of $r\to\infty$ we get for any $\delta(>0)$,

$$\frac{\log M(r,g)}{T(r,g)} \le \frac{3T(2r,g)}{T(r,g)} \le \frac{3(1+\sigma)}{1-\sigma}\ \frac{(2r)^{\lambda_{[g]}+\delta}}{(2r)^{\lambda_{[g]}+\delta-\lambda_{[g]}(2r)}}\frac{1}{r^{\lambda_{[g]}(r)}} \le \frac{3(1+\sigma)}{1-\sigma}(2)^{\lambda_{[g]}+\delta},$$

since $(r)^{\lambda_{[g]}+\delta-\lambda_{[g]}(r)}$ is an increasing function of r.

Since $\sigma,\delta > 0$ are arbitrary, we have

$$\lim_{r\to\infty} inf\ \frac{\log M(r,g)}{T(r,g)} \le 3.2^{\lambda_{[g]}}. \tag{7}$$

Therefore from (4) and (7) we get

$$\liminf_{r\to\infty}\frac{\log^{[n-1]} T(r,f_{[n,g]})}{T(r,g)} \le 3\rho_{[f]}2^{\lambda_{[g]}},\ when\ n\ is\ even$$

Again for even n we have from Lemma 2.8 and (6)

$$\log^{[n-1]} T\left(r, f_{[n,g]}\right) \geq \left(\lambda_{[f]} - \sigma\right) \log M\left(\frac{r}{4^{n-1}}, g\right) + O(1)$$
$$\geq \left(\lambda_{[f]} - \sigma\right) T\left(\frac{r}{4^{n-1}}, g\right) + O(1)$$
$$\geq \left(\lambda_{[f]} - \sigma\right)(1-\sigma)(1+O(1)) \frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{[g]}+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{[g]}+\delta-\lambda_{[g]}\left(\frac{r}{4^{n-1}}\right)}}.$$

Since $(r)^{\lambda_{[g]}+\delta-\lambda_{[g]}(r)}$ is an increasing function of r, we have

$$\log^{[n-1]} T\left(r, f_{[n,g]}\right) \geq \left(\lambda_{[f]} - \sigma\right)(1-\sigma)\left(1+O(1)\right) \frac{(r)^{\lambda_{[g]}(r)}}{(4^{n-1})^{\lambda_{[g]}+\delta}},$$

for all large values of r.
So by (5) for a sequence of values of r tending to infinity

$$\log^{[n-1]} T\left(r, f_{[n,g]}\right) \geq \left(\lambda_{[f]} - \sigma\right) \frac{(1-\sigma)}{(1+\sigma)}\left(1+O(1)\right) \frac{T(r,g)}{(4^{n-1})^{\lambda_{[g]}+\delta}}.$$

Since σ and δ are arbitrary, it follows from the above that

$$\lim_{r\to\infty} sup \frac{\log^{[n-1]} T\left(r, f_{[n,g]}\right)}{T(r,g)} \geq \frac{\lambda_{[f]}}{(4^{n-1})^{\lambda_{[g]}}}, when\ n\ is\ even.$$

Similarly for odd n we get the second part of the theorem.
This proves the theorem.

Theorem 3.2 Let f and g be two non-constant entire functions such that $\lambda_{[f]}$ and $\lambda_{[g]}(>0)$ are finite. If there exist entire functions a_i $(i = 1,2,3,\ldots\ldots\ldots, n; n \leq \infty)$ satisfying $T(r, a_i) = o(T(r,g))$ and $\sum_{i=1}^{n} \delta(a_i, g) = 1$, then

$$\frac{\pi\lambda_{[f]}}{(4^{n-1})^{\lambda_{[g]}}} \leq \lim_{r\to\infty} sup \frac{\log^{[n-1]} T\left(r, f_{[n,g]}\right)}{T(r,g)} \leq \pi\rho_{[f]}, \ \ when\ n\ is\ even.$$

Proof. If $\lambda_{[f]} = 0$ then the first inequality is obvious. Now we suppose that $\lambda_{[f]} > 0$. For $0 < \sigma < \min\{1,\ \lambda_{[f]},\ \lambda_{[g]}\}$ and even n we have from Lemma 2.8, for all large values of r

$$\frac{\log^{[n-1]} T\left(r, f_{[n,g]}\right)}{T(r,g)} \geq \left(1+O(1)\right)\left(\lambda_{[f]} - \sigma\right) \frac{\log M\left(\frac{r}{4^{n-1}}, g\right)}{T(r,g)} + O(1)$$
$$\geq \left(1+O(1)\right)\left(\lambda_{[f]} - \sigma\right) \frac{\log M\left(\frac{r}{4^{n-1}}, g\right)}{T\left(\frac{r}{4^{n-1}}, g\right)} \frac{T\left(\frac{r}{4^{n-1}}, g\right)}{T(r,g)} + O(1) \qquad (8)$$

Also from (5) and (6) we get for a sequence of values of $r \to \infty$ and for any $\delta(>0)$,

$$\frac{T(\frac{r}{4^{n-1}}, g)}{T(r,g)} > \left(1+O(1)\right) \frac{(1-\sigma)}{(1+\sigma)} \frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{[g]}+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{[g]}+\delta-\lambda_{[g]}\left(\frac{r}{4^{n-1}}\right)}} \frac{1}{(r)^{\lambda_{[g]}(r)}}$$
$$\geq \left(1+O(1)\right) \frac{(1-\sigma)}{(1+\sigma)} \frac{1}{(4^{n-1})^{\lambda_{[g]}+\delta}},$$

since $(r)^{\lambda_{[g]}+\delta-\lambda_{[g]}(r)}$ is an increasing function of r.
Since $\sigma,\ \delta > 0$ are arbitrary, therefore using Lemma 2.7 we have from (8)

$$\lim_{r\to\infty} sup \frac{\log^{[n-1]} T\left(r, f_{[n,g]}\right)}{T(r,g)} \geq \frac{\pi\lambda_{[f]}}{(4^{n-1})^{\lambda_{[g]}}}.$$

If $\rho_{[f]} = \infty$, the second inequality is obvious. So we may assume $\rho_{[f]} < \infty$. Then the second inequality follows from Lemma 2.7 and Lemma 2.8.
This proves the theorem.

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