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Complemented Elements in the Lattice of Čech Closure Operators

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ABSTRACT

In this paper the complemented elements in the lattice of all Čech closure operators on a fixed set X are determined. They are precisely the quasi-discrete Čech closure operators on X.

Keywords: Čech closure space, Čech closure operator, quasi-discrete Čech closure operator, lattice of Čech closure operators, complemented element.

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1. INTRODUCTION

A Čech closure operator on a set X is a function $V: P(X) \rightarrow P(X)$ such that

a) $V(\emptyset) = \emptyset$,
b) $A \subset V(A)$, for all $A \in P(X)$,
c) $V(A \cup B) = V(A) \cup V(B)$, for all $A, B \in P(X)$.

where P(X) denotes the power set of X. For brevity, we call V a closure operator on X and (X,V) is called a closure space. Let L(X) denotes the set of all closure operators on X. Then L(X) is a complete lattice with the partial order $\leq$ defined by $V_1 \leq V_2$ if and only if $V_2(A) \subset V_1(A)$ for all $A \in P(X)$.

Let T be a topology on X. Then $V(A) = \overline{A}$, for all $A \in P(X)$, defines a closure operator V on X, called the closure operator associated with the topology T. In this sense a closure space may be regarded as a generalization of a topological space. The lattice of all topologies $\Sigma(X)$ on a fixed set X has been investigated by several authors[3, 6, 8, 9, 10]. Among this,[6] is an interesting survey article in this area.

In[7], the present author investigated the properties of the lattice L(X), comparing it with $\Sigma(X)$, with special reference to complementation. In the present paper we prove that a Closure operator V on a set X is complemented in L(X) if and only if it is quasi-discrete.

In this sequel we need the following

The closure operator D associated with discrete topology is called the discrete closure operator and the closure operator I associated with indiscrete topology is called the indiscrete closure operator. Observe that D is the largest and I is the smallest element in L(X). It can easily be seen that $V_{a,b}$ defined by

$$V_{a,b}(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ X - \{b\} & \text{if } A = \{a\} \\ X & \text{otherwise} \end{cases}$$

for $a, b \in X$, $a \neq b$ is a closure operator on X and the atoms of L(X) are precisely the closure operators of the form $V_{a,b}(A)$. Also note that for a closure operator V on X, $V_{a,b} \leq V$ if and only if $b \notin V(\{a\})$. The dual atoms of L(X) are precisely the closure operators associated with the ultra topologies on X. Recall that the ultra topologies on X, which are the dual atoms in the lattice $\Sigma(X)$ of all topologies on X, can be written in the form $P(X - \{x\}) \cup \mathfrak{U}$ where x is an element of X and $\mathfrak{U}$ is an ultra filter on X which does not contain {x[3]. The closure operator associated with the ultra topology $P(X - \{x\}) \cup \mathfrak{U}$ is given by

$$V(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ A & \text{if } x \in A \text{ or } X - A \in \mathfrak{U} \\ A \cup \{x\} & \text{otherwise} \end{cases}$$

In[10], A. K. Steiner proved that the lattice of topologies $\Sigma(X)$ is complemented. In contrast to this L(X) is complemented if and only if X is finite. Using the fact that L(X) is dually atomistic, it is proved in[7] that no element of L(X) has more than one complement. Multiple complementation in $\Sigma(X)$ is discussed in[8,9] and[13].

2. QUASI-DISCRETE CLOSURE OPERATORS

A topological space X is said to be Alexandroff -discrete if arbitrary intersection of open sets is open. It can be seen that a topological space X is Alexandroff-discrete if and only if $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$ for every arbitrary collection $\{ A_\alpha \}$of subsets of X. A closure operator V on X is called quasi-discrete if $V(\cup A_\alpha) = \cup V(A_\alpha)$ for every arbitrary collection $\{ A_\alpha \}$ of subsets of X (See[2]). It can be seen that a closure operator V on X is quasi-discrete if and only if $V(A) = \bigcup_{a \in A} V(\{a\})$ for every subset of X. When X is finite, every closure operator on X is quasi-discrete. The concept of a quasi-discrete Closure operator is a natural generalization of the concept of an Alexandroff -discrete topology.

Lemma 2.1. Let $V \in L(X)$ be a closure operator with a complement V' in the lattice L(X). Then for any $y \in X$, there exists subsets Y and Y′ of X such that $y \notin V(X\text{-}Y)$, $y \notin V'(X\text{-}Y')$ and $Y \cap Y' = \{y\}$.

Proof: Assume the contrary. Let

$$\mathcal{A} = \{ S \cap S' : y \in S \subset X,\ y \in S' \subset X,\ y \notin V(X - S), y \notin V'(X - S')\}.$$

Then $\mathcal{A}$ is a nonempty family of subsets of X with finite intersection property and $\{y\} \notin \mathcal{A}$. Then there exists an ultrafilter $\mathfrak{U}$ on X containing $\mathcal{A}$ such that $\{y\} \notin \mathfrak{U}$. Let U be the closure operator associated with the ultra topology $P(X - \{y\}) \cup \mathfrak{U}$. Then $V \leq U$. For otherwise there exists a subset B of X such that $y \in U(B)$ and $y \notin V(B)$. Then $X\text{-}B \notin \mathfrak{U}$ and $y \notin B$, for otherwise, U(B)=B. But since

$y \notin V(B)$ and $y \notin V'(X-X) = V'(\emptyset) = \emptyset$, we have that $X-B = (X-B) \cap X \in \mathcal{A} \subset \mathfrak{U}$. This is a contradiction. Therefore $V \leq U$. Similarly $V' \leq U$. Then $V \vee V' \leq U$. This contradicts the fact that V and V' are complements in L(X).Hence the lemma.

Theorem 2.2. A Closure operator V on a set X is complemented in L(X) if and only if it is quasi-discrete.

Proof: Let $V \in L(X)$ be a closure operator with a complement $V' \in L(X)$. Now we will prove that V is quasi-discrete. On the contrary, assume that V is not quasi-discrete. Then there is a non empty subset A of X such that $V(A) \neq \bigcup_{a\in A} V(\{a\})$. Hence there exists $y \in V(A)$ such that $y \notin V(\{a\})$ for every $a \in A$ .Now by Lemma2.1, there exists subsets Y and Y' of X such that $y \notin V(X-Y)$ and $y \notin V'(X-Y')$ and $Y \cap Y' = \{y\}$.

If Y and A are disjoint, then $A \subset X-Y$ and hence $V(A) \subset V(X-Y)$. But $y \in V(A)$ and $y \notin V(X-Y)$, a contradiction. Therefore there exists an $x \in Y \cap A$. Since $x \in A$, $y \notin V(\{x\})$. Thus $V_{x,y} \leq V$. Also $x \notin Y'$, since $Y \cap Y' = \{y\}$ and $x \in Y$. Since $x \in X- Y'$ and $y \notin V'(X-Y')$, we have $y \notin V'(\{x\})$. Then $V_{x,y} \leq V'$. Thus $V_{x,y} \leq V \wedge V'$, a contradiction, since V and V' are complements in L(X). Thus V is quasi-discrete.

Now V be a quasi-discrete closure operator on X. Define $V' : P(X) \to P(X)$ by,

$$V'(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ (X - V(\{a\})) \cup \{a\} & \text{if } A = \{a\} \text{ for some } a \in X \\ \cup_{a\in A} V(\{a\}) & \text{otherwise} \end{cases}$$

for every $A \in P(X)$. Then V' is a quasi-discrete closure operator on X. Also $V \wedge V' = I$, since for every a in X, $(V \wedge V')(\{a\}) = V(\{a\}) \cup V'(\{a\}) = V(\{a\}) \cup (X-V(\{a\})) \cup \{a\} = X$.

To prove that $V \vee V' = D$. Suppose not. Then we obtain $M \subset X$ such that $(V \vee V')(M) \neq M$. Therefore there exists $n \in (V \vee V')(M)$ such that $n \notin M$. Let

$M_1 = \{ x \in M : n \in V(\{x\})\} = \{ x \in M : n \notin V'(\{x\})\}$ and

$M_2 = \{ x \in M : n \notin V(\{x\})\} = \{ x \in M : n \in V'(\{x\})\}$.

Clearly $M = M_1 \cup M_2$. Thus $n \in (V \vee V')(M) = (V \vee V')(M_1) \cup (V \vee V')(M_2)$. Then $n \in (V \vee V')(M_1)$ or $n \in (V \vee V')(M_2)$. If $n \in (V \vee V')(M_1) \subset V'(M_1)$, then $n \in V'(\{x\})$ for some $x \in M_1$, since V' is quasi-discrete. This is a contradiction. If $n \in (V \vee V')(M_2) \subset V(M_2)$, then $n \in V(\{x\})$ for some $x \in M_2$ .This is also a contradiction. Hence the result.

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