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Some Existence Results in Cone B-Metric Spaces

R. A. Siddiqui* and B.P. Tripathi

*Department of Mathematics,
Govt. N.P.G. College of Science, Raipur, Chhattisgarh-492010, INDIA.
email: siddiquirais3@gmail.com.

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ABSTRACT

In this article, we establish some fixed point and common fixed point theorems satisfying a general contractive condition in the setting of cone b-metric spaces with normal solid cone. Also, certain examples are given to support the our results. Our results extend and generalize several known results from the existing literature.

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1. INTRODUCTION

Let X be a non-empty set. A mapping $T: X \to X$ is called a self-map of X. If there is an element $x \in X$ such that T(x) = x, then x is called a fixed point of the self-map T of X. A result giving a set of conditions on T and X under which T has a fixed point is known as a fixed point theorem. In recent times fixed point theorems have gained importance because of their numerous applications. It is well known that the classical Banach contraction principle[4] is the first ever fixed point theorem. The idea of b-metric space was presented by Bokhtin[3] as a generalized form of metric spaces. Bakhtin proves the contraction mapping principle in b-metric space in order to generalize the Banach contraction principle. Moreover, the idea of cone metric space was presented by Haung and Zhang[13]. Their work includes some fixed point results for contractive type mappings in cone metric spaces. In 1993, Cezerwik[6] proved some fixed point theorems in b-metric spaces.

In 2011, Hussain and Shah[14] introduced cone b-metric spaces as a generalization of b-metric spaces over cone. Moreover, Huaping and Shaoyuan in[15], proved some fixed point theorems of contractive mappings with some new examples and applications in cone b-metric spaces. These results improved some fixed point results in metric spaces and b-metric spaces, as well as. They extended other concern work and results of cone metric spaces. Very recently, George *et al*.[11] have introduced the concept of generalized cone b-metric space, which generalizes the concepts of cone metric space, cone rectangular metric space and cone b-metric space. They have proved Banach fixed point theorem and Kannan fixed point theorem in generalized cone b-metric space with solid cone. A generalization of contraction mapping has been introduced and called T-contraction mapping in metric space which is given with another function by Beiranvand[2]. In 2012, Ozavsar and Cevikel[18] introduced the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings in a complete multiplicative metric space. They also gave some topological properties of the relevant multiplicative metric space.

On the other hand, in 2009, Azam *et al*.[1] introduced the concept of cone rectangular metric space. Recently, George *et al*.[11] have introduced the concept of rectangular b-metric space, which is not necessarily Hausdorf and which generalizes the concept of metric space, rectangular metric space and b-metric space. Hxiaoju *et al*.[16] studied common fixed points for weak commutative mappings on a multiplicative metric space. For further details on multiplicative metric space and related concepts, we refer the reader to[5,18]. In this paper, we obtain a unique fixed point and common fixed point theorem for self mapping which satisfy generalized rational contraction mapping on cone b-metric space.

The following definitions and results will be needed in the sequel.

2. PRELIMINARIES

Consistent with Haung and Zhang[13] and Hussain and Shah[14], we will be needed the following definitions in this paper.

Definition 2.1.[13] Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold:

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a, b \in R$, $a\, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$;

(c) $P \cap (-p) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \leq y$ will stand for $y - x \in P0$, where $P0$ stands for the interior of P. If $P0 \neq \phi$ then P is called a solid cone (see[17]).

There exists two kinds of cones: normal cones (with constant K) and non-normal cone[9].

Let E be a real Banach space, $P \subset E$ a cone and $\leq$ partial ordering defined by P. Then P is called normal if there is a number K > 0 such that for all $x, y \in P$,

$$0 \leq x \leq y \text{ implies } ||x|| \leq K ||y|| \qquad (2.1)$$

or equivalently, if for all $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ implies } \lim_{n\to\infty} y_n = x. \tag{2.2}$$

The least positive number K satisfying (2.1) is called the normal constant of P.

Example 2.2.[17] Let $E = C_R^1[0,1]$ with $||x|| = ||x||_\infty + ||x'||_\infty$ on $P = \{x \in P : x(t) > 0\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{t^n}{t}$ and $y_n(t) = \frac{1}{n}$. Then $0 \le x_n \le y_n$ and $\lim_{n\to\infty} y_n = 0$ and $||x_n|| = \max_{t\in[0,1]}\left|\frac{t^n}{n}\right| + \max_{t\in[0,1]}\left|t^{n-1}\right| = \frac{1}{n} > 1$, hence x_n does not converge to zero. It follows by (2.1)
that P is a non-normal cone.

Through this paper we always suppose that E is a real Banach space, P is a cone in E with $\operatorname{int} P \neq 0$ and $\le$ is partial ordering w.r.t cone.

Definition 2.3.[13] Suppose Y be a non-empty set. Let the mapping $d : Y \times Y \to E$ satisfies:

(i) $0 \le d(x, y)$ for all $x, y \in Y$ with $x \neq y$;

(ii)d(x, y) = 0 if and only if x = y;

(iii) d(x, y) = d(y, x) for all $x, y \in Y$;

(iv) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in Y$.

Then the function d is said to be a cone metric on Y and (Y, d) is called a cone metric space.

Definition 2.4.[14] Let that Y be a non-empty set and $s \ge 1$ be a given real number. A function $d : Y \times Y \to E$ is said to be cone b-metric if and only if the following assertions are satisfied:

(i) $0 \le d(x, y)$ for all $x, y \in Y$ with $x \neq y$;

(ii) d(x, y) = 0 if and only if x = y;

(iii) d(x, y) = d(y, x) for $x, y \in Y$;

(iv) $d(x, y) \le s[d(x, z) + d(z, y)]$ for all $x, y, z \in Y$.

Then the function d is said be a cone b-metric on Y and (Y, d) is called a cone b-metric space.

It is to be noted that the class of cone b-metric space is larger than the class of cone metric space, since any cone metric space must be a cone b-metric space.

Therefore, it is obvious that cone b-metric space generalizes b-metric space and cone metric space as well.

We present some examples, which show that introducing a cone b-metric space instead of a cone metric space is meaningful since there exist cone b-metric spaces which are not cone metric spaces.

Example 2.5.[15] Let $X = \{1, 2, 3, 4\}$, $E = R^2$, $P = \{(x, y) \in E : x \ge 0, y \ge 0\}$.

Define $d : X \times X \to E$ by : $d(x, y) = (||x - y||^{-1}, ||x - y||^{-1})$ if $x \neq y$, and $d(x, y) = \theta$ if x = y. Then (X, d) is a cone b-metric space with coefficient $s = \frac{6}{5} > 1$. But it is not a cone metric space since the triangle inequality does not satisfy. We observe that,

$$(1,1) = d(1,2) > d(1,4) + d(4,2) = \left(\frac{1}{3}, \frac{1}{3}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{5}{6}, \frac{5}{6}\right)$$

$$(1,1) = d(3,4) > d(3,1) + d(1,4) = \left(\frac{1}{2}, \frac{1}{2}\right) + \left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{5}{6}, \frac{5}{6}\right).$$

Thus (X, d) is not a cone metric space.

Definition 2.6. Let (X, d) be a metric space. A self mapping $T: X \to X$ is called quasi-contraction if it satisfies the following condition:

$$d(Tx, Ty) \le h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$ and $h \in (0,1)$ is a constant.

Definition 2.7. Let (X, d) be a metric space. A self mapping $T: X \to X$ is called Ciric-quasi-contraction if it satisfies the following condition:

$$d(Tx, Ty) \le h \max\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2} \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

for all $x, y \in X$ and $h \in (0,1)$ is a constant.

The following lemmas will be needed in particular while dealing with cone metric space in which the cone need not be normal.

Lemma 2.8.[10] Let P be a cone and $\{x_n\}$ be a sequence in E such that $c \in \text{int}P$ and $0 \le a_n$ and $a_n \to 0$ when $n \to \infty$, then there exists N such that for all n > N we have $a_n \ll c$.

Lemma 2.9.[10] Let $x, y, z \in E$, , if $x \le y$ and $y \ll z$, then $x \ll z$.

Lemma 2.10.[14] Let P be a cone and $0 \le u \ll c$ for each $c \in \text{int}P$, then u = 0.

Lemma 2.11.[8] Let P be a cone, if $u \in P$ and $u \le tu$ for some $0 \le t < 1$, then u = 0.

Lemma 2.12.[10] Let P be a cone and $a \le b + c$ for each $c \in \text{int}P$, then $a \le b$.

3. MAIN RESULTS

In this section, we shall prove some fixed point theorems using contractive type conditions in the framework of cone b-metric space.

Theorem 3.1. Let (X, d) be a complete cone b-metric space with coefficient $s \le 1$. Suppose that the mapping $T: X \to X$ satisfies the condition

$$d(Tx, Ty) \le \alpha d(x, y) + \beta[d(x, Tx) + d(y, Ty)] + L\max\{d(x, Ty), d(y, Tx)\} \tag{3.1}$$

for all $x, y \in X$, where $\alpha, \beta, L \ge 0$ are constants such that $\beta(\alpha+1)\beta + (\beta^2 + \beta)L < 1$. Then T has unique fixed point in X.

Proof. We choose $x_0 \in X$ and construct the iterative sequence $\{x_n\}$, where $x_n = Tx_{n-1}, n \ge 1$ i.e. $x_{n+1} = Tx_n = T^{n+1}x_0$. From the given condition **(3.1),** we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\le d(x_{n-1}, x_n) + [d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + L\max\{d(x_{n-1};, Tx_n), d(x_n, Tx_{n-1})\} \\
&= d(x_{n-1}, x_n) + [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + L\max\{d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\
&= d(x_{n-1}, x_n) + [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + Ld(x_{n-1}, x_{n+1}) \\
&\le d(x_{n-1}, x_n) + [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + Ls[d(x_{n-1}, x_n) + d(x_n, x_{n+1}] \\
&= (\alpha + \beta + Ls)d(x_{n-1}, x_n) + (\beta + Ls)d(x_n, x_{n+1}],
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1 - \beta - Ls)d(x_n, x_{n+1}) \le (\alpha + \beta + Ls)d(x_{n-1}, x_n) \\
&\Rightarrow d(x_n, x_{n+1}) \le \left(\frac{\alpha + \beta + Ls}{1 - \beta - Ls}\right) d(x_{n-1}, x_n) \qquad (3.2) \\
&\Rightarrow d(x_n, x_{n+1}) \le \mu\, d(x_{n-1}, x_n)
\end{aligned}
$$

where $\mu = \frac{\alpha + \beta + Ls}{1 - \beta - Ls}$. Since $s + (\beta + 1)\beta + (s^2 + s)L < 1$, therefore $\mu < \frac{1}{s}$ Similarly, we obtain

$$d(x_{n-1}, x_n) \le \mu d(x_{n-2}, x_{n-1}) \qquad (3.3)$$

Thus from (3.2) and (3.3), we get

$$d(x_{n-1}, x_n) \le \mu^2 d(x_{n-2}, x_{n-1}).$$

Repeating this process, we obtain

$$d(x_n, x_{n+1}) \le \mu^n d(x_0, x_1). \qquad (3.4)$$

For $m \ge 1$ and $p \ge 1$, we have

$$
\begin{aligned}
d(x_m, x_{m+p}) &\le s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\le sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\le sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) + \dots + s^p d(x_{m+p-1}, x_{m+p}),
\end{aligned}
$$

using (3.4), we get

$$
\begin{aligned}
d(x_m, x_{m+p}) &\le s\mu^m d(x_0, x_1) + s^2\mu^{m+1} d(x_0, x_1) + s^3\mu^{m+2} d(x_0, x_1) + \dots + s^p\mu^{m+p-1} d(x_0, x_1) \\
&\le s^m(1 + s + s^2\mu^2 + s^3\mu^3 + \dots + s^{p-1}\mu^{p-1})d(x_0, x_1) \\
&\le \left(\frac{s\mu^m}{1 - s\mu}\right) d(x_0, x_1)
\end{aligned}
$$

It is obvious that $\left(\frac{s\mu^m}{1 - s\mu}\right) d(x_0, x_1) \to 0$ as $m \to \infty$ for any $p \ge 1$. Let $0 \ll \varepsilon$. Then by using Lemma 1.14, we find that $m_0 \in N$ such that $\left(\frac{s\mu^m}{1 - s\mu}\right) d(x_0, x_1) \ll \varepsilon$. for all $m > m_0$. Thus

$d(x_m, x_{m+p}) \le \left(\frac{s\mu^m}{1-s\mu}\right) d(x_0, x_1) << \varepsilon$ for all $m \ge m_0,\ p \ge 1,$

Implies that $d(x_m, x_{m+p}) << \varepsilon$ for all $m \ge m_0,\ p \ge 1$ So by **lemma 2.9** $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X; d) is a complete cone b-metric space, there exists $x \in X$ such that $x \to u$ as $n \to \infty$. Take $n_0 \in N$ such that $d(x_n, u) << \varepsilon'$ for all $n > n_0$, where $\varepsilon' = \frac{1-s(\beta+Ls)}{s(\alpha+Ls+1)}$.

Hence

$$\begin{aligned}
d(Tx,\ x) &\le s[d(Tx, Tx_n) + d(Tx_n, x)\\
&= d(Tx, Tx_n) + sd(Tx_n, x)\\
&\le s\{d(u, x_n) + [d(x, Tx) + d(x_n, Tx_n) + L\max\{d(u, Tx_n),\ d(x_n, Tx)\}\} + sd(Tx_n, x)\\
&\le sfd(x, x_n) + d(x, Tx) + d(x_n, x_{n+1}) + L\max\{d(x,\ x_{n+1}),\ [d(x_n, x) + d(x, Tx)]\} + d(x_{n+1}, x)\\
&= sd(x, x_n) + sd(x, Tx) + Ls^2[d(x_n, x) + d(x, Tx)] + sd(x_n, x)\\
&= s(\alpha + Ls + 1)d(x, x_n) + (\beta + Ls)d(x, Tx)
\end{aligned}$$

$$\Rightarrow d(Tx, x) << s\frac{(\alpha + Ls + 1)}{1 - s(\beta + Ls)} d(x, x_n)$$

$$\Rightarrow d(Tx, x) << \varepsilon, \text{ for all } n > n_0.$$

Then by **Lemma 2.10**, we obtain that $d(Tx, x) = 0$, that is, $Tx = x$. Thus x is a fixed point of T.

Uniqueness of Fixed Point :

Suppose x^* is another fixed point of T, i.e. $Tx^* = u^*$, then from (i), we have

$$\begin{aligned}
d(x, x) &= d(Tx, Tx^*)\\
&\le d(x,\ x^*) + [d(x, Tx) + d(x^*, Tx^*)] + \max fd(x,\ Tu^*), d(u^*, Tu)\\
&= d(x, x^*) + [d(x,\ x) + d(x^*,\ x^*)] + L\max fd(x,\ x^*),\ d(x, x^*)\}\\
&= \alpha d(x, x^*) + Ld(x, x^*)\\
&= (\alpha + L)d(x, x^*)
\end{aligned}$$

$$\Rightarrow d(x, x^*) \le (\alpha + L)\, d(x, x^*).$$

By **lemma 2.11** if $\alpha + L < 1$, then $d(x, x^*) = 0$, implies $x = x^*$. This completes the proof.

Theorem 3.2. Let (X, d) be a complete cone b-metric space (CCbMS) with coefficient $s \le 1$. Suppose that the mapping $T: X \to X$ satisfies the condition

$$d(T^n x, T^n y) \le \alpha d(x, y) + \beta[d(x, T^n x) + d(y, T^n y)] + L\max\{d(x, T^n y), d(y, T^n x)\} \quad (3.1)$$

for all $x, y \in X$, where $\alpha, \beta, L \ge 0$ are constants such that $\beta(\alpha+1)\beta + (\beta^2 + \beta)L < 1$. Then T has unique fixed point in X.

Proof. From **Theorem 3.1** there exists $u \in X$ such that $T^n u = u$. Then

$$d(Tu,u) = d(TT^n u, T^n u) = d(T^n Tu, T^n u)$$
$$\leq d(Tu,u) + [d(Tu, T^n Tu) + d(u, T^n u)] + L\max\{d(Tu, T^n u), d(u, T^n Tu)\}$$
$$\leq d(Tu,u) + [d(Tu, TT^n u) + d(u, T^n u)] + L\max\{d(Tu, Tnu), \ d(u; TT^n u)\}$$
$$\leq d(Tu,u) + [d(Tu,Tu) + d(u,u)] + L\max\{d(Tu,u), d(u,Tu)\}$$
$$\leq (\alpha + L)d(Tu,u)$$

By **Lemma 2.11**, if $\alpha + L < 1$,, $d(Tu,u) = 0$ and so $Tu = u$. This shows that T has a unique fixed point in X. This completes the proof.

Now, we obtain a common fixed point theorem for two self mappings which satisfy generalized contractive condition in cone b-metric spaces.

Theorem 3.3. Let (X, d) be a complete cone b-metric space with coefficient $s \leq 1$. Suppose that the mapping $S, T : X \to X$ satisfies the condition

$$d(Sx,Ty) \leq \alpha d(x,y) + \beta[d(x,Sx) + d(y,Ty)] + L\max\{d(x,Ty), d(y,Sx)\} \qquad (3.1)$$

for all $x, y \in X$, where $\alpha, \beta, L \geq 0$ are constants such that $\beta(\alpha+1)\beta + (\beta^2 + \beta)L < 1$. Then S and T has unique common fixed point in X.

Proof. Choose $x_0 \in X$. We construct the iterative sequence $\{x_{2n+1}\}$ and $\{x_{2n}\}$ such that $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all $n \geq 0$. From **(3.6)**, we have

$$d(x_{2n+1}, x_{2n}) = d(Sx_{2n}, Tx_{2n-1})$$
$$\leq d(x_{2n}, x_{2n-1}) + [d(x_{2n}, Sx_{2n}) + d(x_{2n-1}, Tx_{2n-1})] + L\max\{d(x_{2n}, Tx_{2n-1}), d(x_{2n-1}, Sx_{2n})\}$$
$$= d(x_{2n}, x_{2n-1}) + [d(x_{2n}, x_{2n+1}) + d(x_{2n-1}. x_{2n})] + L\max\{d(x_{2n}, x_{2n}), d(x_{2n-1}, x_{2n+1})\}$$
$$= d(x_{2n}, x_{2n-1}) + [d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})] + Ld(x_{2n-1}, x_{2n+1})$$
$$\leq d(x_{2n}, x_{2n-1}) + [d(x_{2n}, x_{2n+1}) + d(x_{2n-1}, x_{2n})] + Ls[d(x_{2n}, x_{2n-1}) + d(x_{2n}, x_{2n+1}$$
$$\leq (\alpha + \beta + Ls)d(x_{2n}, x_{2n-1}) + (\beta + Ls)d(x_{2n}, x_{2n+1}))]$$
$$\leq \frac{(\alpha+\beta+Ls)}{1-(\beta+Ls)} d(x_{2n}, x_{2n-1}).$$

Similarly,

$$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$
$$\leq d(x_{2n}, x_{2n+1}) + [d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})] + L\max\{d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\}$$
$$= d(x_{2n}, x_{2n+1}) + [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + L\max\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\}$$
$$= d(x_{2n}, x_{2n+1}) + [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + Ld(x_{2n}, x_{2n+2})$$
$$\leq d(x_{2n}, x_{2n+1}) + [d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + Ls[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]$$
$$\leq (\alpha + \beta + Ls)d(x_{2n}, x_{2n+1}) + (\beta + Ls)d(x_{2n+1}, x_{2n+2})$$
$$\leq \frac{(\alpha+\beta+Ls)}{1-(\beta+Ls)} d(x_{2n}, x_{2n+1}).$$

Inductively, we obtain

$$d(x_{n+1}, x_n) \leq \mu d(x_{n-1}, x_n) \leq \mu\, d(x_{n-2}, x_{n-1}) \leq \ldots \leq \mu^n d(x_0, x_1),$$

where $\mu = \dfrac{(\alpha+\beta+Ls)}{1-(\beta+Ls)}$. Since $s\alpha+(\beta+1)\beta+(s^2+s)L<1$, therefore $\mu<\frac{1}{s}$. Suppose $m, n\geq 1$ and $m> n$, we get

$$\begin{aligned}
d(x_n, x_m) &\leq s[d(x_n, x_{n+1})+d(x_{n+1}, x_m)]\\
&= sd(x_n, x_{n+1})+sd(x_{n+1}, x_m)\\
&\leq sd(x_n, x_{n+1})+s2[d(x_{n+1}, x_{n+2})+d(x_{n+2}, x_m)]\\
&= sd(x_n, x_{n+1})+s^2d(x_{n+1}, x_{n+2})+s^2d(x_{n+2}, x_m)\\
&\leq sd(x_n, x_{n+1})+s^2d(x_{n+1}, x_{n+2})+s^3d(x_{n+2}, x_{n+3})+\ldots+s_{n+m-1}d(x_{n+m-1}, x_m)\\
&\leq s\lambda^n d(x_1, x_0)+s^2\lambda^{n+1}d(x_1, x_0)+s^3\lambda^{n+2}d(x_1, x_0)+\ldots+s^m\lambda^{n+m-1}\\
&= s\lambda^n[1+s\lambda+s^2\lambda^2+s^3\lambda^3+\ldots+(s\lambda)^{m-1}]d(x_1, x_0)\\
&\leq \left[\frac{s\lambda^n}{1-s\lambda}\right]d(x_1, x_0).
\end{aligned}$$

But P is a normal cone with normal constant K, hence we have

$$d(x_n, x_m)\leq\left[\frac{s\lambda^n}{1-s\lambda}\right]||d(x_1, x_0)||.$$

Hence, it implies that $||d(x_n, x_m)||\to 0$ as $n, m\to\infty$ since $0<s\lambda<1$. Therefore $\{x_n\}$ is a Cauchy sequence. Because (X, d) is a complete cone b-metric space, there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$.

Moreover, we have

$$\begin{aligned}
d(z, Tz) &\leq s[d(z, x_{2n+1})+d(x_{2n+1}, Tz)]\\
&= sd(Sx_{2n}, Tz)+sd(z, x_{2n+1})\\
&\leq s[d(x_{2n}, z)+[d(x_{2n}, Sx_{2n})+d(z, Tz)+L\max\{d(x_{2n}, Tz), d(z, Sx_{2n})\}+sd(z, x_{2n+1})\\
&\leq s[d(x_{2n}, z)+[d(x_{2n}, x_{2n+1})]+d(z, Tz)+L\max\{d(x_{2n}, Tz), d(z, x_{2n+1})\}]+sd(z, x_{2n+1})\\
&\leq s[d(x_{2n}, z)+[d(x_{2n}, x_{2n+1})+d(z, Tz)]+Ld(z, x_{2n+1})]+sd(z, x_{2n+1})\\
&\leq sd(x_{2n}, z)+s^2d(x_{2n}, z)+s^2d(x2_{n+1}, z)+sd(z, Tz)]+Lsd(z, x_{2n+1})+sd(z, x_{2n+1})\\
&\leq (s\alpha+s^2\beta)d(x_{2n}, z)+(s+Ls+s^2\beta)d(x_{2n+1}, z)+sd(z, Tz).
\end{aligned}$$

Now using the condition of normal cone, we have

$||d(z, Tz)||\leq K\{(s\alpha+s^2\beta)d(x_{2n}, z)+(s+Ls+s^2\beta)d(x_{2n+1}, z)+sd(z, Tz)\}$.

As $n\to\infty$, we get $||d(z, Tz)||\leq 0$.

Hence $||d(z, Tz)||=0$. So we get $Tz=z$, that is, z is a fixed point of T. In an exactly the same fashion we can prove that $Sz=z$. Hence $Sz=Tz=z$. This shows that z is a unique common fixed point of S and T.

Example 3.4. Let $E = C_R[0,1]$, $P = \{f \in E : f \geq 0\}$, $X = (1, \infty)$ and $d(x, y) = ||x - y||^2\ e^t$. Then (X, d) is a cone b-metric space with the coefficient s = 2. But it is not a cone metric space. We consider the mappings $T: X \to X$ defined by $T(x) = \frac{3x+4}{7}$. Hence

$$
\begin{aligned}
d(Tx, Ty) &= \left| \left(\frac{3x+4}{7}\right) - \left(\frac{3y+4}{7}\right) \right|^2 e^t \\
&= \frac{9}{49} |x - y|^2\ e^t \\
&\leq \frac{1}{2} |x - y|^2\ e^t \\
&= \frac{1}{2} d(x, y).
\end{aligned}
$$

Clearly $1 \in X$ is the unique fixed point of T.

Application 3.5. The object of this section is to apply our result to the mappings involving contraction of integral type to cone b-metric spaces. Denote $\wedge$ the set of functions $\phi: [0, \infty 1) \to [0, \infty 1)$ satisfying the following:

(a) ϕ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$,

(b) for any $\varepsilon > 0$ we have $\int_0^{\varepsilon} \phi(t)dt > 0$.

Theorem 3.6. Suppose (X, d) be a complete cone b-metric space with the coefficient $s > 1$ and P be a normal cone with normal constant K. Assume that the mappings $S, T: X \to X$ satisfy the contraction of integral type:

$$
\int_0^{d(Tx,Ty)} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt + \beta \int_0^{[d(x,Sx)+d(y,ty)} \phi(t)dt + L \int_0^{\max\{d(x,Ty),d(y,Sx)} \phi(t)dt
$$

for all $x\ y \in X$, where $\alpha, \beta, L > 0$ are constants such that $\beta(\alpha+1)\beta + (\beta^2 + \beta)L < 1$ and $\phi \in \wedge$. Then the mappings S and T will have a common fixed point in X.

If we put $\beta = L = 0$ and S = T in Theorem 3.8, we have the following result.

Theorem 3.6. Suppose (X, d) be a complete cone b-metric space with the coefficient $s > 1$ and P be a normal cone with normal constant K. Assume that the mappings $T: X \to X$ satisfy the contraction of integral type:

$$
\int_0^{d(Tx,Ty)} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt
$$

for all $x\ y \in X$, where α with $0 < s\alpha < 1$ and $\phi \in \wedge$. Then T has unique fixed point in X.

Now, we put $\alpha = L = 0$, $\beta = k$ and S = T in Theorem 3.8, we have the following result.

Theorem 3.6. Suppose (X, d) be a complete cone b-metric space with the coefficient $s>1$ and P be a normal cone with normal constant K. Assume that the mappings $T:X\to X$ satisfy the contraction of integral type:

$$\int_0^{d(Tx,Ty)}\phi(t)dt \le k\int_0^{d(x,Ty)+d(y,Tx)}\phi(t)dt$$

for all $x\ y \in X$, where k is a nonnegative real with $k\in[0, 1/(1+s))$ is a constant and $\phi\in \wedge$. Then T has unique fixed point in X.

Putting $\alpha = k,\ \beta = L = 0$ in Theorem 3.1. We get the following result from above theorem:

Corollary 3.9. Let (X, d) be a complete cone b-metric space (CCbMS) with the coefficient s > 1. Suppose that the mapping $T:X\to X$ satisfies:

$$d(Tx, Ty) \le kd(x, y)$$

for all $x, y \in X$, where $k \in (0,1)$ is a constant with sk < 1. Then T has a unique fixed point in X.

Corollary 3.10. Suppose (X, d) be a complete cone b-metric space with the coefficient s > 1. Assume that the mapping $T:X\to X$ satisfies the contractive condition:

$$d(Tx, Ty) \le \beta[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $\beta\in[0, 1/(1+s))$ is a constant. Then T has a unique fixed point in X.

Proof. Proof The proof of Corollary 2.5 immediately follows from Theorem 3.1 by taking $\alpha = L = 0$. This completes the proof.

Remark 3.11. Corollary 3.9 also extends the well known Banach contraction principle[4] to that in the setting of cone b-metric spaces.

Remark 3.12. Corollary 3.10 also extends the Chatterjea contraction[7] to that in the setting of cone b-metric spaces.

CONCLUSION

From the results obtain in this chapter, we conclude to say that it is possible to prove the existence of unique fixed point in a complete cone b-metric space under an specific contractive condition. Also, it is possible to apply the contraction of integral type in cone b-metric space.

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