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ν-theta Function Identities and Discrete Fourier Transform

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ABSTRACT

The extensions of classical identities of Jacobi theta functions are obtained corresponding to the ν-theta functions. These identities are obtained using the properties of eigenvectors corresponding to the discrete Fourier transform in terms of linear combinations of ν-theta functions. In particular these are classical Jacobi theta function identities for $\nu = 1$ corresponding to the $\Phi(2)$.

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1. INTRODUCTION

There are various generalizations of classical Jacobi theta functions in literature starting with the works of Ramanujan and others[1,8]. Matveev[12] has defined ν-theta functions which are periodic and extensions of classical Jacobi theta functions. The classical Jacobi theta functions are the particular case of these ν-theta functions for $\nu = 1$. The eigenvectors of the discrete Fourier transform (DFT) are expressed as a linear combinations in terms of ν- theta functions[12]. This manuscript discusses the identities between ν-theta functions which gives extensions of the identities between classical Jacobi theta functions. The ν-theta functions are not doubly periodic, hence the techniques we use to derive the identities are similar to the one used in[9,10,11]. This technique is different than the methods used in deriving the classical identities which are based on the zeros of Jacobi theta functions or infinite product representation[3]. The basic notations adopted in this paper and some preliminary results are presented in the next section. The identities of theta functions corresponding to the DFT Φ (2) are discussed in the Section 3.

2. PRELIMINARY RESULTS

The matrix $\phi(n)$ corresponding to the DFT of size n is given by

$$\Phi_{jk}(n) \quad = \quad \frac{1}{\sqrt{n}} q^{jk}, \qquad j,k = 0,\cdots,n-1. \qquad q = \frac{2\pi i}{n}. \tag{1}$$

Definition:For $f = (f_0,\cdots,f_{n-1})^t \in C^n$ we define the DFT $\tilde{f} \in C^n$ by $\tilde{f} = \Phi f = (\tilde{f}_0,\tilde{f}_1,\cdots,\tilde{f}_{n-1})$, where

$$\tilde{f}_k \quad = \quad \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_j \, e^{\frac{2\pi i j k}{n}} \; .$$

It is clear from the definition that $\phi^4 = I$. The multiplicities corresponding to the given values of the DFT ϕ (n) are well known and are given by (see[12]).

$$\begin{array}{cccccc} n = 4m+2 & \Rightarrow & m_1 = m & m_2 = m+1 & m_3 = m & m_0 = m+1 \\ n = 4m & \Rightarrow & m_1 = m & m_2 = m & m_3 = m-1 & m_0 = m+1 \\ n = 4m+1 & \Rightarrow & m_1 = m & m_2 = m & m_3 = m & m_0 = m+1 \\ n = 4m+3 & \Rightarrow & m_1 = m+1 & m_2 = m+1 & m_3 = m & m_0 = m+1. \end{array}$$

where m_k is multiplicity of the i^k.

Let τ be complex number with positive imaginary part then generalized υ-theta function is defined by

$$\theta(x,\tau,\nu) = \sum_{m=-\infty}^{m=\infty} exp[\pi i \tau m^{2\nu} + 2\pi i m x]. \qquad \nu \in Z^+, \text{Im}\tau > 0. \tag{2}$$

$\theta(x, \tau, \upsilon)$ is an entire function of x satisfying the relation $\theta(x+1, \tau, \mu) = \theta(x, \tau, \mu)$ and it is periodic function of x . This function satisfies the partial differential equation

$$2(2\pi)^{2\nu-1}(-1)^\nu \frac{\partial\theta}{\partial\tau} = i\frac{\partial^{2\nu}\theta}{\partial x^{2\nu}}. \tag{3}$$

The υ-theta function reduces to the usual theta function for $\upsilon = 1$. The υ-theta function with character a,b is given by $\theta_{a,b}(x, \tau, \upsilon)$

$$\theta_{a,b}(x,\tau,\nu) = \sum_{n\in Z} exp[\pi i \tau (n+a)^{2\nu} + 2\pi i (n+a)(x+b)]. \tag{4}$$

The zeros of υ-theta function is same as the Classical Jacobi theta functions. The Classical Jacobi theta functions are periodic with period 1,τ . The notations in the manuscript are different than the q-series notation in the literature[3]. The classical Jacobi theta functions may be represented by

$$\theta_{0,0}(x,\tau),=\sum_{m=-\infty}^{m=\infty} exp(\pi i m^2\tau+2\pi i m x), \tag{5}$$

$$\theta_{\frac{1}{2},\frac{1}{2}}(x,\tau)=(-1)\sum_{m\in Z} exp[\pi i\tau(m+1/2)^2+2\pi i(m+1/2)(x+1/2)],$$

$$\theta_{\frac{1}{2},0}(x,\tau)=\sum_{m\in Z} exp[\pi i\tau(m+1/2)^2+2\pi i(m+1/2)x],$$

$$\theta_{0,\frac{1}{2}}(x,\tau)=\sum_{m\in Z} exp[\pi i\tau m^2+2\pi i m(x+1/2)].$$

Matveev[12] has proved the following important theorem which will be used in the following sections.

Theorem 2.1. *(Matveev): For any τ with positive imaginary part the vector $v(x,\tau,k,\nu)$ with components $v_j(x,\tau,k,\nu)$, $j=0,1,2,\cdots,n-1$ given by*

$$\begin{aligned} v_j(x,\tau,k,\nu) &= \theta_{\frac{j}{n},0}(x,\tau,\nu)+(-1)^k\theta_{-\frac{j}{n},0}(x,\tau,\nu) \\ &+\frac{1}{\sqrt{n}}\left[(-i)^k\theta\left(\frac{j+x}{n},\frac{\tau}{n^{2\nu}},\nu\right)+(-i)^{3k}\theta\left(\frac{x-j}{n},\frac{\tau}{n^{2\nu}},\nu\right)\right] \quad , \end{aligned} \tag{6}$$

is an eigenvector of the DFT with an eigenvalue i^k :

$$\Phi(n)v(x,\tau,k)=i^k v(x,\tau,k).$$

The proof of the above theorem follows from the fact that $\phi^4 = I$ [12]. The above formula is used to derive the extensions of some of the well known classical identities of the Jacobi theta functions to the generalized υ-theta functions. These identities are explored for the DFT Φ (2) in this manuscript. The extensions of well known fourth order identity and Watson addition formula are obtained.

3. υ-THETA FUNCTIONS AND THE DFT Φ (2)

This section some identities of fourth order are derived between υ-theta functions. These identities are natural extensions of classical well known identities between Jacobi theta functions.

Theorem 3.1. *The generalized fourth order identity between ν-theta functions is given by*

$$\theta^4\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right)-\theta^4\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)=8\theta^3(x,\tau,\nu)\theta_{\frac{1}{2},0}(x,\tau,\nu))+8\theta^3_{\frac{1}{2},0}(x,\tau,\nu)\theta(x,\tau,\nu) \tag{7}$$

Proof: The DFT Φ (2) has only two eigenvalues +1 and -1. The eigenvector corresponding to eigenvalue +1 is given by

$$v(x,\tau,0,\nu)=\begin{bmatrix} 2\theta(x,\tau,\nu)+\sqrt{2}\theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right) \\ 2\theta_{\frac{1}{2},0}(x,\tau,\nu)+\sqrt{2}\theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right) \end{bmatrix}. \tag{8}$$

The eigenvector corresponding to eigenvalue -1 is given by

$$v(x,\tau,2,\nu)=\begin{bmatrix} 2\theta(x,\tau,\nu)-\sqrt{2}\theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right) \\ 2\theta_{\frac{1}{2},0}(x,\tau,\nu)-\sqrt{2}\theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right) \end{bmatrix}. \tag{9}$$

$$\Phi(2)\left[v(x,\tau,0)+v(x,\tau,2)\right]=v(x,\tau,0)-v(x,\tau,2).$$

This gives the following two identities

$$\theta(x,\tau,\nu)+\theta_{\frac{1}{2},0}(x,\tau\nu)=\theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right), \tag{10}$$

$$\theta(x,\tau\nu)-\theta_{\frac{1}{2},0}(x,\tau,\nu)=\theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right). \tag{11}$$

The identities (10), (11) are equivalent to

$$\theta(x,\tau,\nu) = \frac{1}{2}\left[\theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right)+\theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)\right] \tag{12}$$

$$\theta_{\frac{1}{2},0}(x,\tau,\nu) = \frac{1}{2}\left[\theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right)-\theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)\right] \tag{13}$$

The identities (12), (13) gives

$$2\theta^2(x,\tau,\nu)+2\theta^2_{\frac{1}{2},0}(x,\tau,\nu)=\theta^2\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}}\right)+\theta^2\left(\frac{x}{2},\frac{\tau}{2^{2\nu}}\right). \tag{14}$$

Consider the eigenvectors v(x, τ, 0, υ), v(x + 1, τ, 0, υ) corresponding to the eigen value +1. The 2 x 2 minor formed by the determinant of these eigenvectors is zero.

$$\begin{aligned} &\det(v(x,\tau,0,\nu),v(x+1,\tau,0,\nu)=0.\\ - \quad & 4\theta(x,\tau,\nu)\theta_{\frac{1}{2},0}(x,\tau,\nu)+2\sqrt{2}\theta(x,\tau,\nu)\theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right)-2\sqrt{2}\theta_{\frac{1}{2},0}(x,\tau,\nu)\theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right)\\ + \quad & 2\theta^2\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right)-4\theta(x,\tau,\nu)\theta_{\frac{1}{2},0}(x,\tau,\nu)-2\sqrt{2}\theta_{\frac{1}{2},0}(x,\tau,\nu)\theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)\\ - \quad & 2\sqrt{2}\theta(x,\tau,\nu)\theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)-2\theta^2\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)=0. \end{aligned}$$

Using equations (10), (11) the terms with coefficients of $2\sqrt{2}$ cancel each other out. This gives

$$\theta^2\left(\frac{x}{2}, \frac{\tau}{2^{2\nu}}, \nu\right) - \theta^2\left(\frac{x+1}{2}, \frac{\tau}{2^{2\nu}}, \nu\right) = 4\theta(x,\tau,\nu)\theta_{\frac{1}{2},0}(x,\tau,\nu). \tag{15}$$

The identities (14) and (15) leads to the required fourth order identity between υ-theta functions.

Corollary 3.1.1. *The identity (7) at $\nu = 1$ gives the fourth order identity between Jacobi theta functions. This identity in particular gives the well known fourth order identity*

$$\theta^4(0,\tau) - \theta^4_{0,\frac{1}{2}}(0,\tau) = \theta^4_{\frac{1}{2},0}(0,\tau). \tag{16}$$

Proof : At υ = 1 in (7) and replace x by 2x and τ by 4 τ , we get

$$\theta^4(x,\tau) - \theta^4_{0,\frac{1}{2}}(x,\tau) = 8\theta^3(2x,4\tau)\theta_{\frac{1}{2},0}(2x,4\tau)) + 8\theta^3_{\frac{1}{2},0}(2x,4\tau)\theta(2x,4\tau) \tag{17}$$

At the null values we have

$$\theta^4(0,\tau) - \theta^4_{0,\frac{1}{2}}(0,\tau) = 8\theta^3(0,4\tau)\theta_{\frac{1}{2},0}(0,4\tau)) + 8\theta^3_{\frac{1}{2},0}(0,4\tau)\theta(0,4\tau) \tag{18}$$

$$\theta^2(0,2\tau) = \sum_{m,n} exp(\pi i(m^2+n^2)2\tau). \tag{19}$$

Let m + n = n_1 and m-n = n_2, so that n_1 and n_2 are of the same parity. Rewriting the above equation in terms of n_1 and n_2 leads to the following formulae

$$\theta^2(0,2\tau) = \theta^2(0,4\tau) + \theta^2_{\frac{1}{2},0}(0,4\tau). \tag{20}$$

These are well known Landen type transformations. Using the similar argument, we have

$$2\theta(0,4\tau)\theta_{\frac{1}{2},0}(0,4\tau) = \theta^2_{\frac{1}{2},0}(0,2\tau). \tag{21}$$

By (20), (21) in (18), the fourth order identity of theta constants follows. In the next theorem we derive Watson addition formula for υ-theta functions. The classical Watson addition formula follows as a particular case.

Theorem 3.2.

$$\begin{aligned}&2\theta(x_1+x_2,2\tau,\nu)\theta_{\frac{1}{2},0}(x_1-x_2,2\tau,\nu) - 2\theta_{\frac{1}{2},0}(x_1+x_2,2\tau,\nu)\theta(x_1-x_2,2\tau,\nu)\\ &= \theta\left(\frac{x_1+x_2+1}{2}, \frac{\tau}{2}\right)\theta\left(\frac{x_1-x_2}{2}, \frac{\tau}{2^{2\nu-1}}\right) - \theta\left(\frac{x_1-x_2+1}{2}, \frac{\tau}{2^{2\nu-1}}\right)\theta\left(\frac{x_1+x_2}{2}, \frac{\tau}{2^{2\nu-1}}\right).\end{aligned} \tag{22}$$

All the theta functions involved are ν-theta functions.

Proof : Using (10) , (11)

$$\theta\left(\frac{x_1 \pm x_2}{2}, \frac{\tau}{2^{2\nu-1}}, \nu\right) = \theta(x_1 \pm x_2, 2\tau, \nu) + \theta_{\frac{1}{2},0}(x_1 \pm x_2, 2\tau, \nu). \qquad (23)$$

$$\theta\left(\frac{x_1 \pm x_2 + 1}{2}, \frac{\tau}{2^{2\nu-1}}, \nu\right) = \theta(x_1 \pm x_2, 2\tau, \nu) - \theta_{\frac{1}{2},0}(x_1 \pm x_2, 2\tau, \nu). \quad (24)$$

From (8) and (9), we have

$$v(x_1 + x_2, 2\tau, 0, \nu) = \begin{bmatrix} 2\theta(x_1 + x_2, 2\tau, \nu) + \sqrt{2}\theta\left(\frac{x_1+x_2}{2}, \frac{\tau}{2^{2\nu-1}}, \nu\right) \\ 2\theta_{\frac{1}{2},0}(x_1 + x_2, 2\tau, \nu) + \sqrt{2}\theta\left(\frac{x_1+x_2+1}{2}, \frac{\tau}{2^{2\nu-1}}, \nu\right) \end{bmatrix},$$

$$v(x_1 - x_2, 2\tau, 0, \nu) = \begin{bmatrix} 2\theta(x_1 - x_2, 2\tau, \nu) + \sqrt{2}\theta\left(\frac{x_1-x_2}{2}, \frac{\tau}{2^{2\nu-1}}, \nu\right) \\ 2\theta_{\frac{1}{2},0}(x_1 - x_2, 2\tau, \nu) + \sqrt{2}\theta\left(\frac{x_1-x_2+1}{2}, \frac{\tau}{2^{2\nu-1}}, \nu\right) \end{bmatrix}$$

are eigenvectors of Φ (2) corresponding to eigenvalues +1 with multiplicity one . Therefore

$$\det(v(x_1 + x_2, 2\tau, 0, \nu), v(x_1 - x_2, 2\tau, 0, \nu)) = 0.$$

This gives

$$\begin{aligned} &4\theta(x_1 + x_2, 2\tau, \nu)\theta_{\frac{1}{2},0}(x_1 - x_2, 2\tau, \nu) + 2\sqrt{2}\theta(x_1 + x_2, 2\tau, \nu)\theta\left(\frac{x_1 - x_2 + 1}{2}, \frac{\tau}{2^{2\nu-1}}\right) \\ &+2\sqrt{2}\theta_{\frac{1}{2},0}(x_1 - x_2, 2\tau, \nu)\theta\left(\frac{x_1 + x_2}{2}, \frac{\tau}{2^{2\nu-1}}\right) + 2\theta\left(\frac{x_1 + x_2}{2}, \frac{\tau}{2^{2\nu-1}}\right)\theta\left(\frac{x_1 - x_2 + 1}{2}, \frac{\tau}{2^{2\nu-1}}\right) \\ &-4\theta_{\frac{1}{2},0}(x_1 + x_2, 2\tau, \nu)\theta(x_1 - x_2, 2\tau, \nu) - 2\sqrt{2}\theta_{\frac{1}{2},0}(x_1 + x_2, 2\tau, \nu)\theta\left(\frac{x_1 - x_2}{2}, \frac{\tau}{2^{2\nu-1}}\right) \\ &-2\sqrt{2}\theta(x_1 - x_2, 2\tau, \nu)\theta\left(\frac{x_1 + x_2 + 1}{2}, \frac{\tau}{2^{2\nu-1}}\right) - 2\theta\left(\frac{x_1 + x_2 + 1}{2}, \frac{\tau}{2^{2\nu-1}}\right)\theta\left(\frac{x_1 - x_2}{2}, \frac{\tau}{2^{2\nu-1}}\right) = 0. \end{aligned} \qquad (25)$$

In this expression consider the terms with $2\sqrt{2}$ as coefficient. Using formulas (23) and (24) we have

$$
\begin{aligned}
A &= \theta(x_1+x_2,2\tau,\nu)\theta\left(\frac{x_1-x_2+1}{2},\frac{\tau}{2^{2\nu-1}}\right)\\
&= \theta(x_1+x_2,2\tau,\nu)\theta(x_1-x_2,2\tau,\nu)-\theta(x_1+x_2,2\tau,\nu)\theta_{\frac{1}{2},0}(x_1-x_2,2\tau,\nu),\\
B &= \theta_{\frac{1}{2},0}(x_1-x_2,2\tau,\nu)\theta\left(\frac{x_1+x_2}{2},\frac{\tau}{2^{2\nu-1},\nu}\right)\\
&= \theta_{\frac{1}{2},0}(x_1-x_2,2\tau,\nu)\theta(x_1+x_2,2\tau,\nu)+\theta_{\frac{1}{2},0}(x_1+x_2,2\tau,\nu)\theta_{\frac{1}{2},0}(x_1-x_2,2\tau,\nu),\\
C &= \theta_{\frac{1}{2},0}(x_1+x_2,2\tau,\nu)\theta\left(\frac{x_1-x_2}{2},\frac{\tau}{2^{2\nu-1}}\right)\\
&= \theta_{\frac{1}{2},0}(x_1+x_2,2\tau,\nu)\theta(x_1-x_2,2\tau,\nu)+\theta_{\frac{1}{2},0}(x_1+x_2,2\tau,\nu)\theta_{\frac{1}{2},0}(x_1-x_2,2\tau,\nu),\\
D &= \theta(x_1-x_2,2\tau,\nu)\theta\left(\frac{x_1+x_2+1}{2},\frac{\tau}{2^{2\nu-1}}\right)\\
&= \theta(x_1-x_2,2\tau,\nu)\theta(x_1+x_2,2\tau,\nu)-\theta(x_1-x_2,2\tau,\nu)\theta_{\frac{1}{2},0}(x_1+x_2,2\tau,\nu).
\end{aligned}
$$

It is clear that A+B-C-D = 0. Hence all the terms with coefficients $2\sqrt{2}$ cancel each other out. Equation (25) becomes

$$
\begin{aligned}
&4\theta(x_1+x_2,2\tau)\theta_{\frac{1}{2},0}(x_1-x_2,2\tau)-4\theta_{\frac{1}{2},0}(x_1+x_2,2\tau)\theta(x_1-x_2,2\tau)\\
&=2\theta\left(\frac{x_1+x_2+1}{2},\frac{\tau}{2}\right)\theta\left(\frac{x_1-x_2}{2},\frac{\tau}{2^{2\nu-1}}\right)-2\theta\left(\frac{x_1-x_2+1}{2},\frac{\tau}{2^{2\nu-1}}\right)\theta\left(\frac{x_1+x_2}{2},\frac{\tau}{2^{2\nu-1}}\right)
\end{aligned}
$$

This proves (22).

Theorem 3.3. *(Watson addition formula)*

$$
\begin{aligned}
\theta_{\frac{1}{2},\frac{1}{2}}(x_1,\tau)\theta_{\frac{1}{2},\frac{1}{2}}(x_2,\tau) &= \theta(x_1+x_2,2\tau)\theta_{\frac{1}{2},0}(x_1-x_2,2\tau)\\
&- \theta(x_1-x_2,2\tau)\theta_{\frac{1}{2},0}(x_1+x_2,2\tau) \qquad (26)
\end{aligned}
$$

Proof: At υ = 1 in (22) the identity reduces the identity between Jacobi theta function. The complete details of the proof can be seen in[9].

The next theorem we give Riemann's identity for At υ-theta functions. The υ -theta functions satisfies the following identity. All theta functions involved are υ-theta functions.

Theorem 3.4. *(Riemann identity for ν-theta functions.)*

$$
\begin{aligned}
&4\theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)+4\theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta(u,\tau)\theta(v,\tau)\\
&+4\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\theta(x,\tau)\theta(y,\tau)+4\theta(x,\tau)\theta(y,\tau)\theta(u,\tau)\theta(v,\tau)\\
&=\theta\left(\frac{x}{2},\frac{\tau}{4}\right)\theta\left(\frac{y}{2},\frac{\tau}{4}\right)\theta\left(\frac{u}{2},\frac{\tau}{4}\right)\theta\left(\frac{v}{2},\frac{\tau}{4}\right)\\
+&\ \theta\left(\frac{x+1}{2},\frac{\tau}{4}\right)\theta\left(\frac{y+1}{2},\frac{\tau}{4}\right)\theta\left(\frac{u}{2},\frac{\tau}{4}\right)\theta\left(\frac{v}{2},\frac{\tau}{4}\right)\\
+&\ \theta\left(\frac{x}{2},\frac{\tau}{4}\right)\theta\left(\frac{y}{2},\frac{\tau}{4}\right)\theta\left(\frac{u+1}{2},\frac{\tau}{4}\right)\theta\left(\frac{v+1}{2},\frac{\tau}{4}\right)\\
+&\ \theta\left(\frac{x+1}{2},\frac{\tau}{4}\right)\theta\left(\frac{y+1}{2},\frac{\tau}{4}\right)\theta\left(\frac{u+1}{2},\frac{\tau}{4}\right)\theta\left(\frac{v+1}{2},\frac{\tau}{4}\right).
\end{aligned}
\qquad (27)
$$

Proof : The υ-theta identities from (10), (11) we have

$$\theta(x,\tau,\nu)+\theta_{\frac{1}{2},0}(x,\tau,\nu) = \theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right), \quad (28)$$

$$\theta(y,\tau,\nu)+\theta_{\frac{1}{2},0}(y,\tau,\nu) = \theta\left(\frac{y}{2},\frac{\tau}{2^{2\nu}},\nu\right), \quad (29)$$

$$\theta(x,\tau,\nu)-\theta_{\frac{1}{2},0}(x,\tau,\nu) = \theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right), \quad (30)$$

$$\theta(y,\tau,\nu)-\theta_{\frac{1}{2},0}(y,\tau,\nu) = \theta\left(\frac{y+1}{2},\frac{\tau}{2^{2\nu}},\nu\right), \quad (31)$$

These set of equations gives

$$\begin{aligned}&\theta(x,\tau,\nu)\theta(y,\tau,\nu)+\theta_{\frac{1}{2},0}(x,\tau,\nu)\theta_{\frac{1}{2},0}(y,\tau,\nu)\\ = &\ \theta\left(\frac{x}{2},\frac{\tau}{2^{2\nu}},\nu\right)\theta\left(\frac{y}{2},\frac{\tau}{2^{2\nu}},\nu\right)\\ + &\ \theta\left(\frac{x+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)\theta\left(\frac{y+1}{2},\frac{\tau}{2^{2\nu}},\nu\right) \quad (32)\end{aligned}$$

Changing the variables (x, y) to (u, v) in (32)

$$\begin{aligned}&\theta(u,\tau,\nu)\theta(v,\tau,\nu)+\theta_{\frac{1}{2},0}(u,\tau,\nu)\theta_{\frac{1}{2},0}(v,\tau,\nu)\\ = &\ \theta\left(\frac{u}{2},\frac{\tau}{2^{2\nu}},\nu\right)\theta\left(\frac{v}{2},\frac{\tau}{2^{2\nu}},\nu\right)\\ + &\ \theta\left(\frac{u+1}{2},\frac{\tau}{2^{2\nu}},\nu\right)\theta\left(\frac{v+1}{2},\frac{\tau}{2^{2\nu}},\nu\right) \quad (33)\end{aligned}$$

Using (32), (33), we get the required formula.
In the next result we show the Riemann's identity between Jacobi theta functions as the particular case of (27).

Theorem 3.5. *At $\nu = 1$ the identity (27) gives in particular the Riemann's identity given by*

$$\begin{aligned}&\theta_{\frac{1}{2},\frac{1}{2}}(x,\tau)\theta_{\frac{1}{2},\frac{1}{2}}(y,\tau)\theta_{\frac{1}{2},\frac{1}{2}}(u,\tau)\theta_{\frac{1}{2},\frac{1}{2}}(v,\tau)+\theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau)\\ &\theta_{0,\frac{1}{2}}(u,\tau)\theta_{0,\frac{1}{2}}(v,\tau)\theta_{0,\frac{1}{2}}(x,\tau)\theta_{0,\frac{1}{2}}(y,\tau)+\theta(x,\tau)\theta(y,\tau)\theta(u,\tau)\theta(v,\tau)\\ = &\ 2\sum_{m,n,p,q\in\frac{1}{2}Z} exp\left[\pi i\left(m^2+n^2+p^2+q^2\right)\tau+2\pi i\left(mx+ny+pu+qv\right)\right]. \quad (34)\end{aligned}$$

$m,\ n,\ p,\ q$ are either integers or $m,\ n,\ p,\ q$ are $\in\ \frac{1}{2}+Z$ and $\sum m = m+n+p+q\in 2Z$.

Proof : At υ = 1 in (27) the identity reduces to the identity between Jacobi theta functions. The complete details of the proof are given in[10].

For simplicity if we do a change of the variable as follows

$$\begin{aligned} n_1 &= \frac{1}{2}(n+m+p+q), & x_1 &= \frac{1}{2}(x+y+u+v) \\ m_1 &= \frac{1}{2}(n+m-p-q), & y_1 &= \frac{1}{2}(x+y-u-v) \\ p_1 &= \frac{1}{2}(n-m+p-q), & u_1 &= \frac{1}{2}(x-y+u-v) \\ q_1 &= \frac{1}{2}(n-m-p+q), & v_1 &= \frac{1}{2}(x-y-u+v). \end{aligned}$$

Then the particular restrictions on parameters n; m; p; q of the summation above exactly means that the resulting n_1, m_1, p_1, q_1 are integers. Also observe that we have the identities :

$$\sum n^2 = \sum n_1^2 \quad \text{and} \quad \sum xn = \sum x_1 n_1$$

The equation (34) can be written as

$$\begin{aligned} & \theta_{\frac{1}{2},\frac{1}{2}}(x,\tau)\theta_{\frac{1}{2},\frac{1}{2}}(y,\tau)\theta_{\frac{1}{2},\frac{1}{2}}(u,\tau)\theta_{\frac{1}{2},\frac{1}{2}}(v,\tau) + \theta_{\frac{1}{2},0}(x,\tau)\theta_{\frac{1}{2},0}(y,\tau)\theta_{\frac{1}{2},0}(u,\tau)\theta_{\frac{1}{2},0}(v,\tau) \\ & \theta_{0,\frac{1}{2}}(u,\tau)\theta_{0,\frac{1}{2}}(v,\tau)\theta_{0,\frac{1}{2}}(x,\tau)\theta_{0,\frac{1}{2}}(y,\tau) + \theta(x,\tau)\theta(y,\tau)\theta(u,\tau)\theta(v,\tau) \\ = \; & 2\theta(x_1,\tau)\theta(y_1,\tau)\theta(u_1,\tau)\theta(v_1,\tau). \end{aligned} \tag{35}$$

The beautiful account of identities derived from Riemann identity (35) are given in[2].

This includes all the fourth order identities of theta functions. This shows that all these identities have an extensions in the form of υ- theta functions. The above method shows that they can be derived using the techniques discussed in the manuscript.

CONCLUSION

This manuscript has discussed the identities of υ-theta functions corresponding to the DFT Φ(2). The classical identities of the Jacobi theta functions corresponds to the υ = 1. It would be interesting to study these identities corresponding to υ = 2. There is a natural extensions of these identities for the DFT Φ(3) as discussed in[9,11].

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