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Einstein Finslerian Space with Special (α, β)-Metric

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ABSTRACT

Einstein-Finslerian metrics are solutions to Einstein eld equation in General Relativity containing Ricci-flat metrics. In this paper, we have determined the Riemann curvature of special (α , β)-metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$; where μ_1, μ_2 and μ_3 are constants. Then we have obtained the necessary and sufficient condition for that (α , β)-metric to be Einstein metric, when β is a constant killing form. Finally, we have proved that the mentioned Einstein metric is Ricci flat.

MSC: 53B20, 53B40, 53C60.

Keywords: (α , β)-metrics, Riemannian curvature, Ricci curvature, Einstein Finsler space.

1. INTRODUCTION

In Finslerian space, (α, β)-metrics are the special class of Finslerian metrics which are having a major role in formulating applications in Einstein theory of relativity, mechanics, biology, control theory, etc.[1, 2, 3, 4]. In 2003, D. Bao and Robles have shown Einstein Randers metric is necessarily Ricci constant[5]. In[6], Einstein Schur type lemma for $(\alpha, \beta)-$metrics proved by the authors B. Rezaei, A. Razavi and N. Sadeghzadeh. In general theory of relativity, the solutions of Einstein field equations obtained by Einstein metrics. It is necessary to compute the Riemann and Ricci curvature (α, β)-metrics in order to characterize Einstein-Finsler (α, β)-metric. On a Finsler manifold M, Riemannian curvature $R_y : T_xM \rightarrow T_xM$ is $R_y(u) =$

$R^i_k(y)u^k \frac{\partial}{\partial x^i}$. Then, Ricci scalar defined as $(x,y) = R^i_k$. If the Ricci scalar is of the form $Ric = c(x)F^2(x,y)$ for some function $c(x)$ on Manifold $M,$ then Finsler metric is Einstein. i.e., the Ricci scalar is a function of x alone. A manifold is called Ricci flat if Ricci tensor vanishes, which represents vacuum solution to Einstein field equations in relativity[5, 7].

In[6], Razaei, Razavi and Sadeghazadeh have considered generalized Kropina metric, Matsumoto metric with a constant Killing form and obtained the necessary and sufficient conditions to be Einstein metrics. In[8], Rafie and Rezaei proved that the second Schur type lemma for Finsler-Matsumoto metric. Then,[9] X. Cheng, Z. Shen and Y. Tian, proved (α,β)-metric is Ricci flat. In[10], projectively related Einstein Finsler metrics are classified over compact manifold. In[11,12], the Einstein criterion is obtained for special Finslerian (α,β)-metrics.

2. PRELIMINARIES

For a Finsler space, geodesic spray is given below

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where spray coefficients G^i is

$$G^i(x,y) = \frac{1}{4} g^{ij}(x,y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\} y^j y^k, \quad (2.1)$$

The coefficients G^i_j, G^i_{jk} of spray G^i defined as

$$G^i_j = \frac{\partial G^i}{\partial y^j},$$

$$G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

In Finsler geometry, Riemannian curvature tensor R_y is the function $R_y = R^i_k(y)dx^k \otimes \frac{\partial}{\partial x^i}|x : T_xM \to T_xM$ is defined as,

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^i \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \quad (2.2)$$

Suppose $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric, then $R^i_k = R^i_{jkl}(x)y^j y^l$, where $R^i_{jkl}(x)$ denote the co-efficients of Riemannian curvature tensor. Thus, R_y in (2.2) is called Riemannian curvature in Finsler geometry. With respect to the Riemannian curvature, Ricci scalar function for the Finsler metric defined by $\rho = \frac{1}{L^2} R^i_i$, which is positive homogeneous function of degree 0 in y. It shows that $\rho(x,y)$ dependent on the direction of the flag pole y, but not on its length. Then Ricci tensor is given by

$$Ric_{ij} = \left\{\frac{1}{2} R^i_j\right\} y^i y^j. \quad (2.3)$$

Suppose the Ricci tensor on a manifold becomes zero, then such manifold called as Ricci-flat[5]. The Ricci tensor plays major role in Finsler geometry to study the Einstein criterion for Finsler spaces. If the Ricci scalar is a function of x-only, then Finsler metric becomes Einstein metric. i.e.,

$$Ric_{ij} = \rho(x) g_{ij}\,, \tag{2.4}$$

In[13], M. Matsumoto given an expression for G^i as

$$2G^i = \gamma^i_{00} + 2B^i, \tag{2.5}$$

where

$$B^i = \left(\frac{E}{\propto}\right) y^i = \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0 \; - \; \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha} C \left\{ \left(\frac{y^i}{\alpha}\right) - \left(\frac{\alpha}{\beta}\right) b^i \right\}\right),$$

$$E = \left(\frac{\beta L_\beta}{L}\right) C, \qquad C = \frac{\alpha\beta(r_{00} L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

$$b^i = a^{ir} b_r\,, \qquad b^2 = b^r b_r\,, \qquad \gamma^2 = b^2\alpha^2 - \beta^2,$$

$$r_{ij} = \frac{1}{2}\left(b_{i|j} + b_{j|i}\right), \qquad s_{ij} = \frac{1}{2}\left(b_{i|j} - b_{j|i}\right),$$

$$s^i_j = a^{ih} s_{hj},\; s_{j\,=}\, b_i s^i_j. \tag{2.6}$$

Where h-covariant derivation with respect to the Riemannian connection in the space (M, α) is denoted by " | " and (a^{ij}) is inverse of (a_{ij}). The Christoffel symbols in the space (M, α) are denoted by the functions γ^i_{ik}. Now (2.6) becomes

$$B^i = \{\tilde{p} r_{00} + \tilde{q}_0 s_0) y^i + \tilde{r} s^i_0 + (\tilde{s}_0 L_{00} - \tilde{t} s_0) b^i, \tag{2.7}$$

Where

$$\tilde{p} = \beta\left(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}\right)/2L(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}),$$

$$\tilde{q} = -\alpha\beta L_\beta\left(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha}\right)/L L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}),$$

$$\tilde{r} = \alpha L_\beta / L_\alpha,$$

$$\tilde{s}_0 = \alpha^3 L_{\alpha\alpha}/2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}),$$

$$\tilde{t} = -\alpha^4 L_{\alpha\alpha} L_\beta / L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}). \tag{2.8}$$

Substituting (2.7) in (2.5) and (2.2), we get Berwald's formula for Riemannian curvature tensor as follows

$$R^i_k(y) = \bar{R}^i_k + \{2B^i_{|k} - y^j\left(B^i_{|k}\right)^k_y - \left(B^i\right)^j_y\left(B^j\right)^k_y + 2B^j\left(B^i\right)_{y^j y^k}\}. \tag{2.9}$$

If $r_{ij} = 0\left(s_{ij} = 0\right)$, then 1-form β is said to be Killing (closed) 1-form. If β is Killing vector, then it is said to be a constant Killing form and it has constant length with respect to α, equivalently $r_{ij} = 0,\; s_i = 0$.

3. RIEMANNIAN CURVATURE OF FINSLERIAN SPACE WITH SPECIAL $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-METRIC

In this section, we derive Riemannian curvature for Finslerian space with special (α, β) −metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, where μ_1 , μ_2 and μ_3 are constants.

For $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$, we have

$$\left.\begin{aligned} L_\alpha &= \mu_1 - \mu_3 \frac{\beta^2}{\alpha^2}, \\ L_\beta &= \mu_2 + 2\mu_3 \frac{\beta}{\alpha}. \end{aligned}\right\} \tag{3.1}$$

Now by using values of (3.1), equation (2.8) becomes

$$\tilde{r} = \frac{\mu_2\alpha^3+2\mu_3\alpha^2\beta}{\mu_1\alpha^2-\mu_3\beta^2}. \tag{3.2}$$

If β is a constant Killing form, then by substituting (3.2) in (2.7), we get

$$B^i = \frac{\mu_2\alpha^3+2\mu_3\alpha^2\beta}{\mu_1\alpha^2-\mu_3\beta^2} s_0^i. \tag{3.3}$$

Now, by covariant and contravariant differentiation of (3.3), we obtain

$$B^i_{.j} = \frac{A_1 y_j}{(\mu_1\alpha^2-\mu_3\beta^2)^2} s_0^i + \frac{A_2 b_j}{(\mu_1\alpha^2-\mu_3\beta^2)^2} s_0^i + \frac{\mu_2\alpha^3+2\mu_3\alpha^2\beta}{\mu_1\alpha^2-\mu_3\beta^2} s_j^i, \tag{3.4}$$

$$B^i_{|j} = \frac{A_2 b_{0|j}}{(\mu_1\alpha^2-\mu_3\beta^2)^2} s_0^i + \frac{\mu\alpha^3+2\mu\alpha^2\beta}{\mu\alpha^2-\mu_3\beta^2} s^i_{0|j}, \tag{3.5}$$

where

$$A_1 = \mu_1\mu_2\alpha^3 - 3\mu_2\mu_3\alpha\beta^2 - 4\mu_3^2\beta^3,$$

$$A_2 = 2\mu_1\mu_3\alpha^4 + 2\mu_3^2\alpha^2\beta^2 + 2\mu_2\mu_3\alpha^3\beta.$$

From (3.4), we have

$$B^i B^i_{.j.i} = 0\,, \tag{3.6}$$

$$B^i_{.j} B^j_{.i} = \frac{2A_1(\mu_2\alpha^3+2\mu_3\alpha^2\beta)}{(\mu_1\alpha^2-\mu_3\beta^2)^3} s_0^i s_{0i} + \frac{(\mu_2\alpha^3+2\mu_3\alpha^2\beta)^2}{(\mu_1\alpha^2-\mu_3\beta^2)^2} s_{ij} s^{ij}. \tag{3.7}$$

Differentiating (3.5) with respect to y^i and then transvecting by y^j , we have

$$y^j\left(B^i_{|j}\right)_{.i} = 0\,. \tag{3.8}$$

Finally by substituting (3.4) to (3.8) in (2.9), we obtain

$$R^i_i = \bar{R}^i_i + \left\{2B^i_{|i} - y^j\left(B^i_{|j}\right)_{.i} - \left(B^i\right)_{.j}\left(B^j\right)_{.i} + 2B^j\left(B^i\right)_{y^j y^i}\right\}$$

$$\begin{aligned} = \bar{R}^i_i &+ \frac{(2\mu_1\mu_2^2 - 4\mu_1^2\mu_3)\alpha^6 - 6\mu_2^2\mu_3\alpha^4\beta^2 - 16\mu_2\mu_3^2\alpha^3\beta^3 - 12\mu_3^3\alpha^2\beta^4}{(\mu_1\alpha^2-\mu_3\beta^2)^3} s_0^i s_{i0} \\ &+ \frac{2(\mu_2\alpha^3+2\mu_3\alpha^2\beta)}{(\mu_1\alpha^2-\mu_3\beta^2)} s^i_{0|i} - \frac{\mu_2^2\alpha^6 + 4\mu_3^2\alpha^4\beta^2 + 4\mu_2\mu_3\alpha^5\beta}{(\mu_1\alpha^2-\mu_3\beta^2)^2} s^{ij} s_{ij}, \end{aligned} \tag{3.9}$$

where R^i is the Riemannaian curvature of the Finsler space.
Thus we have

Theorem 3.1. *Let F be a Finsler space with special (α,β)-metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$. Suppose β is a constant killing form, then the Riemannain curvature of the Finsler space is given in the equation (3.9).*

4. EINSTEIN CRITERION FOR FINSLERIAN SPACE WITH SPECIAL (α,β)-METRIC

Here, we establish the Einstein criterion for the metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$; where

μ_1, μ_2 and μ_3 are constants. A Finsler metric $L = L(x, y)$ on M^n is called an Einstein metric if the Ricci scalar satisfies the following condition

$$Ric(n-1)\lambda L^2, \tag{4.1}$$

Where $\lambda = \lambda(x)$ is a scalar on M. L is Ricci constant, if λ is constant. If L is Einstein, then we have $L^2\ Ric(x) = R^i_i$.

Hence, we have

Theorem 4.2. Let F^n is a Finsler space with special (α, β)-metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$; where μ_1, μ_2 and μ_3 are constants and β is a constant killing form. Then F^n is Einstein if and only if the Ricci scalar is of the form $Rat + \alpha Irrat = 0$, where both Rat and $Irrat$ are given in equation (4.4)

Proof: By using the Riemannian curvature is given in (3.9), we get the Ricci curvature as follows

$$\overline{Ric}_{00} + \frac{(2\mu_1\mu_2^2 - 4\mu_1^2\mu_3)\alpha^6 - 6\mu_2^2\mu_3\alpha^4\beta^2 - 16\mu_2\mu_3^2\alpha^3\beta^3 - 12\mu_3^3\alpha^2\beta^4}{(\mu_1\alpha^2 - \mu_3\beta^2)^3} s_0^i s_{i0} + \frac{2(\mu_2\alpha^3 + 2\mu_3\alpha^2\beta)}{(\mu_1\alpha^2 - \mu_3\beta^2)} s_{0|i}^i - \frac{\mu_2^2\alpha^6 + 4\mu_3^2\alpha^4\beta^2 + 4\mu_2\mu_3\alpha^5\beta}{(\mu_1\alpha^2 - \mu_3\beta^2)^2} s^{ij}s_{ij} + \frac{(\mu_1\alpha^2 + \mu_2\alpha\beta + \mu_3\beta^2)^2}{\alpha^2} Ric(x) = 0. \tag{4.2}$$

Multiplying (4.2) by $\alpha^2(\mu_1\alpha^2 - \mu_3\beta^2)^3$ removes y from the denominators and after simplifications, we get

$$[\mu_1^3\alpha^8 - 3\mu_1^2\mu_3\alpha^6\beta^2 + 3\mu_1\mu_3^2\alpha^4\beta^4 - \mu_3^3\alpha^2\beta^6]\overline{Ric}_{00} + [(2\mu_1\mu_2^2 - 4\mu_1^2\mu_3)\alpha^8 - 6\mu_2^2\mu_3\alpha^6\beta^2 - 16\mu_2\mu_3^2\alpha^5\beta^3 - 12\mu_3^3\alpha^4\beta^4]s_0^i s_{i0} + [-\mu_1\mu_2^2\alpha^{10} - 4\mu_1\mu_2\mu_3\alpha^9\beta + (\mu_2^2\mu_3 - 4\mu_1\mu_3^2)\alpha^8\beta^2 + 4\mu_2\mu_3^2\alpha^7\beta^3 + 4\mu_3^3\alpha^6\beta^4]s^{ij}s_{ij} + [2\mu_1^2\mu_2\alpha^9 + 4\mu_1^2\mu_3\alpha^8\beta - 4\mu_1\mu_2\mu_3\alpha^7\beta^2 - 8\mu_1\mu_3^2\alpha^6\beta^3 + 2\mu_2\mu_3^2\alpha^5\beta^4 + 4\mu_3^3\alpha^4\beta^5]s_{0|i}^i + \left[-\mu_1^5\alpha^{10} - 2\mu_1^4\mu_2\alpha^9\beta - (\mu_1^3\mu_2^2 - \mu_1^4\mu_3)\alpha^8\beta^2 + 4\mu_1^3\mu_2\mu_3\alpha^7\beta^3 + (\mu_1^2\mu_2^2\mu_3 + 2\mu_1^3\mu_2^2)\alpha^6\beta^4 - (2\mu_1^2\mu_3^3 + 3\mu_1\mu_2^2\mu_3^2)\alpha^4\beta^6 - 4\mu_1\mu_2\mu_3^3\alpha^3\beta^7 - (\mu_1\mu_3^4 - \mu_2^2\mu_3^3)\alpha^2\beta^8 + 2\mu_2\mu_3^4\alpha\beta^9 + \mu_3^5\beta^{10}\right]Ric(x) = 0. \tag{4.3}$$

Now we have to characterize the Einstein criterion for the (α, β)-metric, thus we classify both rational and irrational terms from the above equation, thus we have

$$Rat + \alpha Irrat = 0,$$

Where Rat and $Irrat$ obtained in the following cases:

$$\begin{aligned} Rat = {} & [\mu_1^3\alpha^8 - 3\mu_1^2\mu_3\alpha^6\beta^2 + 3\mu_1\mu_3^2\alpha^4\beta^4 - \mu_3^3\alpha^2\beta^6]\overline{Ric}_{00} \\ & + [(2\mu_1\mu_2^2 - 4\mu_1^2\mu_3)\alpha^8 - 6\mu_2^2\mu_3\alpha^6\beta^2 - 12\mu_3^3\alpha^4\beta^4]s_0^i s_{i0} \\ & + [-\mu_1\mu_2^2\alpha^{10} + (\mu_2^2\mu_3 - 4\mu_1\mu_3^2)\alpha^8\beta^2 + 4\mu_3^3\alpha^6\beta^4]s^{ij}s_{ij} \\ & + [4\mu_1^2\mu_3\alpha^8\beta - 8\mu_1\mu_3^2\alpha^6\beta^3 + 4\mu_3^3\alpha^4\beta^5]s_{0|i}^i \\ & + \left[-\mu_1^5\alpha^{10} - (\mu_1^3\mu_2^2 - \mu_1^4\mu_3)\alpha^8\beta^2 + (\mu_1^2\mu_2^2\mu_3 + 2\mu_1^3\mu_2^2)\alpha^6\beta^4 \right. \\ & \left. - (2\mu_1^2\mu_3^3 + 2\mu_1\mu_2^2\mu_3^2)\alpha^4\beta^6 - (\mu_1\mu_3^4 - \mu_2^2\mu_3^3)\alpha^2\beta^8 + \mu_3^5\beta^{10}\right]Ric(x) \end{aligned}$$

$$Irrat = [-16\mu_2\mu_3^2\alpha^4\beta^3]s_0^i s_{i0} + [4\mu_2\mu_3^2\alpha^6\beta^3 - 4\mu_1\mu_2\mu_3\alpha^8\beta]s^{ij}s_{ij} + [2\mu_1^2\mu_2\alpha^8 - 4\mu_1\mu_2\mu_3\alpha^6\beta^2 + 2\mu_2\mu_3^2\alpha^4\beta^4]s_{0|i}^i + [-2\mu_1^4\mu_2\alpha^8\beta + 4\mu_1^3\mu_2\mu_3\alpha^6\beta^3 - 4\mu_1\mu_2\mu_3^3\alpha^2\beta^7 +$$

$2\mu_2\mu_3^4\beta^9]Ric(x)$ (4.4)

Let $Rat = P(y)$ and $Irrat = Q(y)$. We know that, the quadratic $\alpha^2 = a_{ij}(x)y^iy^j$ would have been factored into linear term. Otherwise α can never be polynomial in y. Then hyperplane with zero set contradicts the positive definiteness of a_{ij}. Now, suppose the expression *Rat* is not zero. Then (4.4) is the product of $Irrat$ with a non-polynomial factor α, which is not possible.
So $Rat = 0$ and also $Irrat = 0$ becuase α is positive at all $y \neq 0$.
Hence the proof.
If L is Einstein, then $Rat = 0$. Then by equation (4.4) we have

$$C_1\alpha^2 + C_2 = 0 \tag{4.5}$$

where C_1 and C_2 are as follows

$$\begin{aligned}C_1 = {} & [\mu_1^3\alpha^6 - 3\mu_1^2\mu_3\alpha^4\beta^2 + 3\mu_1\mu_3^2\alpha^2\beta^4 - \mu_3^3\beta^6]\overline{Ric}_{00} \\ & + [(2\mu_1\mu_2^2 - 4\mu_1^2\mu_3)\alpha^6 - 6\mu_2^2\mu_3\alpha^4\beta^2 - 12\mu_3^3\alpha^2\beta^4]s_0^is_{i0} \\ & + [-\mu_1\mu_2^2\alpha^8 + (\mu_2^2\mu_3 - 4\mu_1\mu_3^2)\alpha^6\beta^2 + 4\mu_3^3\alpha^4\beta^4]s^{ij}s_{ij} \\ & + [4\mu_1^2\mu_3\alpha^6\beta - 8\mu_1\mu_3^2\alpha^4\beta^3 + 4\mu_3^3\alpha^2\beta^5]s_{0|\mathrm{i}}^i \\ & + \big[-\mu_1^5\alpha^8 - (\mu_1^3\mu_2^2 - \mu_1^4\mu_3)\alpha^6\beta^2 + (\mu_1^2\mu_2^2\mu_3 + 2\mu_1^3\mu_2^2)\alpha^4\beta^4 \\ & - (2\mu_1^2\mu_3^3 + 3\mu_1\mu_2^2\mu_3^2)\alpha^2\beta^6 - (\mu_1\mu_3^4 - \mu_2^2\mu_3^3)\beta^8\big]Ric(x)\end{aligned}$$

$$C_2 = \big[\mu_3^5\beta^{10}\big]Ric(x)$$

Thus, by (4.5) it can be seen that α^2divides C_2. So $\beta = 0$.
Thus, we state that the following

Theorem 4.3. *Suppose Finsler metric $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ with constant killing form β, is Einstein metric then it is Ricci flat.*

Example 1. *If $\mu_3 = 1$, then the metric becomes: $L = \mu_1\alpha + \mu_2\beta + \frac{\beta^2}{\alpha}$. According to equations (4.4) this metric Einstein if it satisfies,*
$Rat + \alpha Irrat = 0$,
Where

$$\begin{aligned}Rat = {} & [\mu_1^3\alpha^8 - 3\mu_1^2\alpha^6\beta^2 + 3\mu_1\alpha^4\beta^4 - \alpha^2\beta^6]\overline{Ric}_{00} \\ & + [(2\mu_1\mu_2^2 - 4\mu_1^2)\alpha^8 - 6\mu_2^2\alpha^6\beta^2 - 12\alpha^4\beta^4]s_0^is_{i0} \\ & + [-\mu_1\mu_2^2\alpha^{10} + (\mu_2^2 - 4\mu_1)\alpha^8\beta^2 + 4\alpha^6\beta^4]s^{ij}s_{ij} \\ & + [4\mu_1^2\alpha^8\beta - 8\mu_1\alpha^6\beta^3 + 4\alpha^4\beta^5]s_{0|\mathrm{i}}^i \\ & + \big[-\mu_1^5\alpha^{10} - (\mu_1^3\mu_2^2 - \mu_1^4)\alpha^8\beta^2 + (\mu_1^2\mu_2^2 + 2\mu_1^3\mu_2^2)\alpha^6\beta^4 \\ & - (2\mu_1^2 + 3\mu_1\mu_2^2)\alpha^4\beta^6 - (\mu_1 - \mu_2^2)\alpha^2\beta^8 + \beta^{10}\big]Ric(x),\end{aligned}$$

$$\begin{aligned}Irrat = {} & [-16\mu_2\alpha^4\beta^3]s_0^is_{i0} + [4\mu_2\alpha^6\beta^3 - 4\mu_1\mu_2\alpha^8\beta]s^{ij}s_{ij} \\ & + [2\mu_1^2\mu_2\alpha^8 - 4\mu_1\mu_2\alpha^6\beta^2 + 2\mu_2\alpha^4\beta^4]s_{0|\mathrm{i}}^i \\ & + [-2\mu_1^4\mu_2\alpha^8\beta + 4\mu_1^3\mu_2\alpha^6\beta^3 - 4\mu_1\mu_2\alpha^2\beta^7 + 2\mu_2\beta^9]Ric(x)\,.\end{aligned}$$

5. CONCLUSION

In this paper, by the characteristic condition we have proved Finslerian space with $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ (where μ_1, μ_2 and μ_3 are constants) to be Einstein. Further, under the assumption that length of β with respect to α is constant, we have shown that the Finslerian metric with $L = \mu_1\alpha + \mu_2\beta + \mu_3\frac{\beta^2}{\alpha}$ (where μ_1, μ_2 and μ_3 are constants) is an Einstein metric if and only if β is parallel with to α and α is Ricci flat.

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