Journal of Computer and Mathematical Sciences, *Vol.10(3),631-634 March 2019* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Idempotents of M_2 (Z_{21}[x])

Poonam[1] and Shilpa Aggrwal[2]

[1]Department of Mathematics,
Govt. College, Hisar, INDIA.
[2]Department of Mathematics, KUK.

(Received on: March 15, 2019)

ABSTRACT

In this article we will discuss the idempotents of M_2 (Z_{21}[x]).

Keywords: Idempotents, polynomials, trace etc.

INTRODUCTION

Idempotents in ring theory plays a vital role in the ring theory.

Here, we will find the idempotents in matrix ring M_2 (Z_{21}[x]) where Z_{21}[x] is the poly nominal ring over the ring Z_{21}. For any ring R, I (R) will denote for set of all idempotents in R. For any positive integer n, M_n (R) will denote the ring of n×nmatrices over a ring R.
For a 2×2 matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$over a commutative ring R, determinate of A is ad-bc and trace of A is a+d.

Definition: Let R be a ring. An element a ϵ R is said to be idempotent in R if $a^2 = a$

Theorem 1 If R is a commutative ring then I(R[x]) = I (R)

Theorem 2 Any non trivial idempotent is M_2 (Z_{21} [x]) is one of the following form

1 $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$,$\begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$

2 $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{bmatrix}$, where a(x), b(x), c(x) ϵ Z_{21}[x] such that a(x){1-a(x)} = b(x) c(x).

3 $\begin{bmatrix} 7a(x) & 7b(x) \\ 7c(x) & 7(1-a(x)) \end{bmatrix}$,

where a(x), b(x), c(x) are polynomial in Z_{21}[x] such that a(x) (1-a(x))-b(x) c(x) = 3f(x) for some f(x) ϵ Z_{21}[x]

4 $\begin{bmatrix} 15a(x) & 15b(x) \\ 15c(x) & 5(3+2a(x)) \end{bmatrix}$, where a(x) (5+a(x)) –5b(x)c(x) = 7g(x) for some g(x) $\epsilon Z_{21}[x]$

5 $\begin{bmatrix} 3 & 7\mathrm{b(x)} \\ 7\mathrm{c(x)} & 10 \end{bmatrix} or \begin{bmatrix} 10 & 7\mathrm{b(x)} \\ 7\mathrm{c(x)} & 3 \end{bmatrix}$ where 9-7b(x)c(x)=15

6 $\begin{bmatrix} 17 & 7\mathrm{b(x)} \\ 7\mathrm{c(x)} & 17 \end{bmatrix}$ where 16-7b(x)c(x)=15

7 $\begin{bmatrix} 1+7a(x) & 7\mathrm{b(x)} \\ 7\mathrm{c(x)} & 15+15a(x) \end{bmatrix}$where a(x), b(x), c(x) are polynomials in $Z_{21}[x]$ such that (1+7a(x)) (15+15a(x)) – 7b(x) 5c(x) = 15.

8 $\begin{bmatrix} 1+3a(x) & 3b(x) \\ 3c(x) & 7-3a(x) \end{bmatrix}$, where a(x), b(x), c(x) are polynomial in $Z_{21}[x]$ such that (1+3a(x)) (7-3a(x)) – 9b(x) c(x)= 7

Proof: As the idempotents in $Z_{21}[x]$ are the idempotents in Z_{21}. Therefore the idempotents in$Z_{21}[x]$ are 0, 1, 7, 15 let A=$\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$be a non trivial idempotent of M_2 (Z_{21} [x]). For our convenience, we will take a(x) = a, b(x) = b, c(x) = c, d(x)=d

Now A is idempotent, so $a^2+bc=a$, b(a+d)=b,c(a+d)=c and $bc+d^2=d$. Also determinant of A is an idempotent inZ_{21}so the determinant of A is 0 or 1 or 7 or 15.

If determinant of A is 1 Then A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, a trivial idempotent inM_2 (Z_{21} [x]). Hence the determinant of A is 0 or 7 or 15. Also, trace of A is in Z_{15} i.e. a+d ϵZ_{21}.

Case 1: Determinant of A is 0 i.e. ad-bc = 0

Since A is an idempotent

Therefore, $a^2+bc+bc+d^2 = a^2+2bc+d^2 = a^2+2ad+d^2 = (a+d)^2$

Thus (a+d) is an idempotent inZ_{21} .Thus (a+d) is either 0 or 1 or 7 of 15.

If a+d=0,then we get A to be a zero matrix, which is trivial idempotent in M_2 (Z_{21} [x])

If a+d=1 then d=1-a and hence ad-bc=0 gives $a^2+bc=a$, (a+d) b=b. Also (a+d) c=c and $bc+d^2=1-a$. Thus $A^2 = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$. Thus in this case, matrix A= $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{bmatrix}$,where a(x), b(x), c(x) ϵ $Z_{21}[x]$ such that a(x){1-a(x)} = b(x) c(x).

If a+d=7gives d=7-a and hence ad-bc = 0 gives $a^2+bc=7a$ and so, 6a=0. Also (a+d) b=b implies 6b=0 and (a+d)c=c implies 6c=0

Therefore, $a=7a'(x)$, $b=7b'(x)$ and $c=7c'(x)$, where $a'(x)$, $b'(x)$, $c'(x)$ are polynomial in $Z_{21}[x]$

Since ad-bc=0, we get $7a'(x)(7-7a'(x))=7b'(x)7c'(x)$, which is equivalent to $a'(x)(1-a'(x))-b'(x)c'(x)=3f(x)$ for some polynomial f(x) $\epsilon Z_{21}[x]$

Hence A =$\begin{bmatrix} 7a(x) & 7b(x) \\ 7c(x) & 7(1-a(x)) \end{bmatrix}$, where a(x), b(x), c(x) are polynomial in $Z_{21}[x]$ such that a(x) (1-a(x))-b(x) c(x) = 3f(x) for some f(x) $\epsilon Z_{21}[x]$

If a+d=15 then d=15-a

Now ad-bc = 0 gives $a^2+bc=15a$. Thus 14a=0

Also (a+d) b=b gives 14b=0 and (a+d) c=c gives 14c=0
Therefore, a=15a'(x), b=15b'(x) and c=15c'(x) and d= 15+6a'(x), where a'(x), b'(x) and c'(x) are polynominal in $Z_{21}[x]$
Now since ad-bc = 0, we get 15a'(x) (15+6a' (x)) = 15b' (x)15 c' (x) which is equivalent to a'(x) (5+2a' (x)) - 5b' (x)5 c' (x) = 7 g(x) for some g(x)ϵ$Z_{21}[x]$.

hence, idempotent matrix is $A = \begin{bmatrix} 15a(x) & 15b(x) \\ 15c(x) & 5(3+2a(x)) \end{bmatrix}$, where a(x) (5+a(x)) –5b(x)c(x) = 7g(x) for some g(x) ϵ$Z_{21}[x]$.

Case 2: Determinant of A is 7. This means ad-bc=7.
So, we get $a^2+bc+bc+d^2 = a^2+2(ad-7)+d^2 = (a+d)^2-14 = (a+d)^2+7$
Trace of matric A is idempotent if a+d=8 or 64
If a+d=8 then ad-bc=7implies 7a=7 i.e. a=1+3a(x)for some polynomial a(x) in $Z_{21}[x]$.
Also (a+d)b =b gives 7b=0 and (a+d) c=c gives 7c=0
i.e. b=3b(x) and c=3c(x) for some polynomials b(x) and c(x) in $Z_{15}[x]$.

Hence matrix $A = \begin{bmatrix} 1+3a(x) & 3b(x) \\ 3c(x) & 7-3a(x) \end{bmatrix}$, where a(x), b(x), c(x) are polynomial in $Z_{21}[x]$ such that (1+3a(x)) (7-3a(x)) – 9b(x) c(x)= 7
If a+d =14 then d=14-a
Now ad-bc=7 gives 13a=7 i.e a=7
Also b(a+d)=b gives 13b=0 i.e. b=0 and c(a+d)=c gives 13c=0i.e.c=0
Thus a=7 b=0 c=0 d=7
Hece matrix $A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Case 3: Determinate of A is 15 i.e. ad-bc=15
Since A is idempotent, therefore $a^2+bc+bc+d^2 = a^2+2(ad-15)+d^2 = (a+d)^2-30 = (a+d)^2+12$
Trace of matrix A is idempotent iffa+d=6 or 9or 13 or16
If a+d=6 then ad-bc=15 implies 5a=15
Also b(a+d)=b gives 5b=0 i.e. b=0 and c(a+d)=c gives 5c=0 i.e. c=0
Thus a=3, b=0, c=0 and d=3
Hence, matrix $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ which is not possible as determinant of A is 15
If a+d=9 then ad-bc=15 implies 8a=15 i.e. a=15
Also b(a+d)=b gives 8b=0 i.e. b=0 and c(a+d)=c gives 8c=0 i.e. c=0
Thus a=15, b=0, c=0 and d=15
Hence, matrix $A = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$.
If a+d=13 then ad-bc=15 implies 12a=15 i.e. a=3 or 10 or 17
Also b(a+d)=b gives 12b=0 i.e. b=0 and c(a+d)=c gives 12c=0 i.e. c=0
Therefor b=7b(x) and c=7c(x) for some polynomials b(x) and c(x) in$Z_{21}[x]$.
Hence matrix $A = \begin{bmatrix} 3 & 7b(x) \\ 7c(x) & 10 \end{bmatrix}$ or $A = \begin{bmatrix} 10 & 7b(x) \\ 7c(x) & 3 \end{bmatrix}$ where 9-7b(x)c(x)=15 and

A= $\begin{bmatrix} 17 & 7b(x) \\ 7c(x) & 17 \end{bmatrix}$ where 16-7b(x)c(x)=15

If a+d=16 then ad-bc=15 implies 15a=15 i.e. a=1 or 8 or 15 i. e. a=1+7a(x) for some polynomial a(x) in $Z_{21}[x]$.

Also b(a+d)=b gives 15b=0 i.e. b=7b(x) and c(a+d)=c gives 15c=0 i.e. c=7c(x) for some polynomials b(x) and c(x) in$Z_{21}[x]$.

Hence matrix A= $\begin{bmatrix} 1 + 7a(x) & 7b(x) \\ 7c(x) & 15 + 15a(x) \end{bmatrix}$where a(x), b(x), c(x) are polynomials in $Z_{21}[x]$ such that (1+7a(x)) (15+15a(x)) – 7b(x) 5c(x) = 15.

REFERENCES

1. “Contemporary Abstract Algebra” by Joseph A. Gallian. ISBN-1305887859, 9781305887855.
2. “A Course in Abstract Algebra” by V.K. Khanna& S.K. Bhambri, Second Edition,Vikas Publication House Pvt Limmited,1999,070698675X, 9780706986754.
3. “A first Court in Abstract Algebra” (7th Edition) by – J.B. Fraleigh. ISBN - 13:9780201763904.
4. “Abstract Algebra” (3rd Edition) by David S. Dummit, Richard M.Foote,ISBN – 9780471433347.
5. “Abstract Algebra” by I.N. Hersterin. ISBN-10:0471368792.