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# Convolution Integral Equation Involving Generalized Hypergeometric Function and *H*-function of Two Variables

**Poonam Kumari and Yashwant Singh***

Department of Mathematics,
Institute of Integrated & Honors Studies,
Kurukshetra University, Kurukshetra, INDIA.
*Department of Mathematics,
Government College, Kaladera, Jaipur, Rajasthan, INDIA.
email: pkumari@kuk.ac.in, dryashu23@yahoo.in.

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**ABSTRACT**

In the present paper, the authors have established a solution regarding convolution integral equation whose kernel is a generalized hypergeometric function ${}_PF_Q[.]$ and the $H$ -function of two variables. Some interesting special cases of main result have also been discussed.

**2010 Mathematical subject classification:** 33C99.

**Keywords:** Convolution Integral Equation**,** Laplace Transform, Convolution Theorem, H-function of Two Variables, Generalized Hypergeometric Function.

## 1. INTRODUCTION

If *f(t)* and *g(t)* are piecewise continuous function on $[0,\infty)$, then the convolution integral of *f(t)* and *g(t)* is,

$$(f * g)(t) = \int_0^2 f(t-\tau)g(\tau)d\tau$$

A nice property of convolution integrals is.

$$(f * g)(t) = (g * f)(t)$$

Or

$$\int_0^2 f(t-\tau)g(\tau)d\tau = \int_0^2 f(t-\tau)g(\tau)d\tau$$

The following fact will allow us to take the inverse transforms of a product of transforms.

$$L(f*g)=F(s)G(s)$$

$$L^{-1}\{F(s)G(s)\}=(f*g)(t)$$

Generalized hypergeometric function is defined as:

$$ {}_PF_Q\left[\ a_P\ ;\ b_Q\ ;z\right]={}_PF_Q\left[\frac{a_P}{b_Q};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{P}(a_j)_n}{\prod_{j=1}^{Q}(b_j)_n}\frac{z^n}{n!}, \qquad (1.1)$$

Where for brevity, $(a_P)$ denotes the array of parameters $a_1,\ldots,a_P$ with similar interpretation for $(b_Q)$ etc. . For further details one can refer Rainville[5].

The $H$-function of two variables (Mittal and Gupta[3], p.172) using the following notation, which is due essentially to Srivastava and Panda ( [7], p.266, eq. (1.5) et seq.)is defined and represented as:

$$H[x,y]=H\begin{bmatrix}x\\y\end{bmatrix}=H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(\xi,\eta)\,\theta_2(\xi)\,\theta_3(\eta)\,x^{\xi}y^{\eta}d\xi\,d\eta \qquad (1.2)$$

Where

$$\phi(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)} \qquad (1.3)$$

$$\theta_2(\xi)=\frac{\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)} \qquad (1.4)$$

$$\theta_3(\eta)=\frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)} \qquad (1.5)$$

## 2. THE CONVOLUTION INTEGRAL EQUATION

The solution of the following convolution integral equation has been given:

$$\int_0^x t^{\rho-1}(x-t)^{\sigma-1}\,{}_PF_Q\left[\ g_P\ ;\ h_Q\ ;at^u(x-t)^\eta\right].$$
$$H^{o,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} t(x-t) \\ t(x-t)\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right] f(t)dt = g(x) \tag{2.1}$$

Where Re(σ)>0, Re(η)>0 and

(i) $R=\sum_{j=1}^{p_1}\alpha_j+\sum_{j=1}^{p_2}\gamma_j-\sum_{j=1}^{q_1}\beta_j-\sum_{j=1}^{q_2}\delta_j<0$

(ii) $S=\sum_{j=1}^{p_1}A_j+\sum_{j=1}^{p_2}E_j-\sum_{j=1}^{q_1}B_j-\sum_{j=1}^{q_2}F_j<0$

(iii) $U=-\sum_{j=n_1+1}^{p_1}\alpha_j-\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_2}\gamma_j-\sum_{j=n_2+1}^{p_2}\gamma_j>0$

(iv) $V=-\sum_{j=n_1+1}^{p_1}A_j-\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{m_3}F_j-\sum_{j=m_3+1}^{q_3}F_j+\sum_{j=1}^{n_3}E_j-\sum_{j=n_3+1}^{p_3}E_j>0$

(V) $|\arg x|<\frac{1}{2}U\pi$ (vi) $|\arg y|<\frac{1}{2}V\pi$

**Solution:** In order to solve (2.1), we first take Laplace transform of both sides of (2.1), we get

$$\int_0^\infty e^{-px}\{\int_0^x t^{\rho-1}(x-t)^{\sigma-1}\,{}_PF_Q\left[\ g_R\ ;\ h_S\ ;at^u(x-t)^\eta\right]$$
$$H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} t(x-t) \\ t(x-t)\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right] f(t)\,dt\ \ dx=\int_0^\infty e^{-px}g(x)dx$$
$$f(r)H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} 1 \\ 1\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right].$$
$$\int_0^\infty e^{-px}\{\int_0^x t^{\rho+ur+\xi+\zeta-1}(x-t)^{\sigma+\eta r+\xi+\zeta-1}f(t)\,dt\ \ dx=\overline{g}(p)$$

Changing the order of integration, we get

$$f(r)H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} 1 \\ 1\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right].$$
$$\int_0^\infty t^{\rho+ur+\xi+\zeta-1}f(t)\{\int_t^\infty e^{-px}(x-t)^{\sigma+\eta r+\xi+\zeta-1}\,dx\ \ dt=\overline{g}(p)$$

Putting (x-t)=u, we obtain

$$f(r)H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right].$$

$$\int_0^\infty t^{\rho+ur+\xi+\zeta-1}f(t)\{\int_0^\infty e^{-p(u+t)}u^{\sigma+\eta r+\xi+\zeta-1}\,du\ \ dt=\overline{g}(p)$$

$$f(r)H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right].$$

$$\int_0^\infty e^{-pt}t^{\rho+ur+\xi+\zeta-1}f(t)\{\int_0^\infty e^{-pu}u^{\sigma+\eta r+\xi+\zeta-1}\,du\ \ dt=\overline{g}(p)$$

$$f(r)H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}1\\1\end{matrix}\middle|\begin{matrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right].$$

$$(-1)^{\rho+ur+\xi+\zeta-1}\frac{d^{\rho+ur+\xi+\zeta-1}}{dp^{\rho+ur+\xi+\zeta-1}}\overline{f}(p)\frac{\Gamma(\sigma+\eta r+\xi+\zeta)}{p^{\sigma+\eta r+\xi+\zeta}}=\overline{g}(p)$$

$$\frac{f(r)H_2\left[p^{-1},p^{-1}\right]}{p^{\sigma+\eta r}}\{(-1)^{\rho+ur+\xi+\zeta-1}\frac{d^{\rho+ur+\xi+\zeta-1}}{dp^{\rho+ur+\xi+\zeta-1}}\overline{f}(p)\}=\overline{g}(p)$$

Or $\{\frac{d^{\rho+ur+\xi+\zeta-1}}{dp^{\rho+ur+\xi+\zeta-1}}\overline{f}(p)\}=\frac{(-1)^{-\rho-ur-\xi-\zeta+1}}{f(r)H_2\left[p^{-1},p^{-1}\right]}p^{\sigma+\eta r}\overline{g}(p)$

(2.2)

Where $\overline{f}(p)$ and $\overline{g}(p)$ denote the Laplace transform of f(t) and g(t), respectively, and

$$H_2\left[p^{-1},p^{-1}\right]=H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1+1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}p^{-1}\\p^{-1}\end{matrix}\middle|\begin{matrix}(1-\sigma-\eta r,1,1),(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

(2.3)

And $f(r)=\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}(g_j)_r}{\prod_{j=1}^{q}(h_j)_r}\frac{a^r}{r!}$. (2.4)

A series expansion for $H_2\left[p^{-1},p^{-1}\right]$ can be obtain as a special case of series expansion of H[x,y] ([6],eq.(6.2.1),p.84)and, since this specialization leads to a power series, the series representation for the reciprocal can be formed. To do this we note that

$$H_2\left[p^{-1},p^{-1}\right]=\sum_{M,N=0}^{\infty}C_{M,N}(-p)^{-M-N}\frac{\Gamma(\sigma+\eta r+M+N)}{M!N!} \quad (2.5)$$

Where $C_{M,N}=\phi(M,N)\theta_2(M)\theta_3(N)$ (2.6)

Where $\phi(\xi,\zeta),\theta_2(\xi),\theta_3(\zeta)$ are given by (1.3), (1.4), (1.5) respectively.

From the well -known rearrangement property (Rainville[5], p.56)

$$\sum_{M,N=0}^{\infty} F(M,N) = \sum_{M=0}^{\infty}\sum_{N=0}^{M} f(M-N,N) \tag{2.7}$$

We can rewrite $H_2$ as a single (power) series in the form

$$H_2\left[p^{-1},p^{-1}\right] = \sum_{\nu=0}^{\infty} h_\nu p^{-\nu} \tag{2.8}$$

Where $$h_\nu = \frac{(-1)^\nu \Gamma(\sigma+\eta r+\nu)}{\nu!}\sum_{\mu=0}^{\nu} {}^{\nu}_{\mu}\ C_{\nu-\mu,\mu} \tag{2.9}$$

If k denotes the least value of ν for which $h_\nu \neq 0$, then

$$H_2\left[p^{-1},p^{-1}\right] = p^{-k}\sum_{n=0}^{\infty} h_{k+n} p^{-n} \tag{2.10}$$

So that if we let the coefficients $H_\lambda$ be determined by the relation

$$\left[\sum_{n=0}^{\infty} h_{k+n} p^{-n}\right]^{-1} = \sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda} \tag{2.11}$$

Then (2.2) becomes

$$\frac{d^{\rho+ur+\xi+\zeta-1}}{dp^{\rho+ur+\xi+\zeta-1}}\overline{f}(p) = (-1)^{-\rho-ur-\xi-\zeta+1} p^{\sigma+\eta r+k}\overline{g}(p)\frac{1}{f(r)}\sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda}$$

$$= (-)^{-\rho-ur-\xi-\zeta+1} p^{-(\rho-k-\sigma-\eta r)}\left[\frac{1}{f(r)}\sum_{\lambda=0}^{\infty} H_\lambda p^{-\lambda}\right]\left[p^\rho \overline{g}(p)\right] \tag{2.12}$$

Consequently, on taking the inverse Laplace transform of (2.12) and applying its convolution theorem, we obtain the following:

**Theorem:** If Re(σ)>0, Re(η)>0, $g^r(0)=0$ for 0≤r≤ρ,ρ an integer, Re(ρ-k-σ-ηr)>0, then under suitable restrictions on the parameters of the H-functions of two variables occurring in (2.1) [obtainable easily from the set of conditions (i) to (vi) mentioned with (2.1)]the solution to the convolution integral equation (2.1) is given by

$$(-1)^{\rho+ur+\xi+\zeta-1} t^{\rho+ur+\xi+\zeta-1} f(t) = (-1)^{-\rho-ur-\xi-\zeta+1}\int_0^t (t-x)^{\rho-k-\sigma-\eta r-1} V(t-x) g^\rho(x)dx$$

Or $$t^{\rho+ur+\xi+\zeta-1} f(t) = \int_0^t (t-x)^{\rho-k-\sigma-\eta r-1} V(t-x) g^\rho(x)dx \tag{2.13}$$

Where

$$V(x) = \frac{1}{f(r)}\sum_{\lambda=0}^{\infty}\frac{H_\lambda x^\lambda}{\Gamma(\rho-k+\lambda-\sigma-\eta r)} \tag{2.14}$$

The coefficients $H_\lambda$ being defined by the recurrences

$$H_k H_0=1, \text{ and for } \mu>0 \text{ by } \sum_{\lambda=0}^{\mu} H_\lambda h_{\mu+k-\lambda}=0 \tag{2.15}$$

And the power series coefficients $h_\nu$ being given by (2.9).

## 3. SPECIAL CASES

(i) If we take $p_1=q_1=0$ in (2.1), the H-function of two variables reduces to the product of two single Fox's H-function as:

$$\int_0^x t^{\rho-1}(x-t)^{\sigma-1}\, {}_P F_Q\left[\; g_P \;;\; h_Q \;; at^u(x-t)^\eta\right] H_{p_2,q_2}^{o,n_2}\left[t(x-t)\left|\begin{smallmatrix}(c_j,\gamma_j)_{1,p_2}\\(d_j,\delta_j)_{1,q_2}\end{smallmatrix}\right.\right] H_{p_3,q_3}^{o,n_3}\left[t(x-t)\left|\begin{smallmatrix}(e_j,E_j)_{1,p_3}\\(f_j,F_j)_{1,q_3}\end{smallmatrix}\right.\right] f(t)dt=g(x) \tag{3.1}$$

Where $g^{(r)}(0)=0$ for $0\le r\le\rho$, $\rho$ an integer and $Re(\sigma)>0$, $Re(\eta)>0$, has its solution given by

$$t^{\rho+ur+\xi+\zeta-1} f(t)=\int_0^t (t-x)^{\rho-k-\sigma-\eta r-1} V(t-x) g^{\rho}(x)dx \tag{3.2}$$

Where $Re(\rho-k-\sigma-\eta r)>0$ and

$$\omega(x)=\frac{1}{f(r)}\sum_{\lambda=0}^{\infty}\frac{H_\lambda x^\lambda}{\Gamma(\rho-k+\lambda-\sigma-\eta r)} \tag{3.3}$$

The $H_\lambda$ determined by (2.15), $h_\nu$ given by (2.9), and the coefficients in (2.6) reduces to the form:

$$C_{\nu-\mu,\mu}=\theta_2(\nu-\mu)\theta_3(\mu)$$

(ii) If we put r=0 in (2.1), we get the result due to Buschman, Koul and Gupta[1] in some slight different form:

$$\int_0^x t^{\rho-1}(x-t)^{\sigma-1} H_{p_1,q_1:p_2,q_2:p_3,q_3}^{o,n_1:m_2,n_2:m_3,n_3}\left[\begin{smallmatrix}(x-t)\\(x-t)\end{smallmatrix}\left|\begin{smallmatrix}(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{smallmatrix}\right.\right] f(t)dt=g(x)$$

Where $g^{(r)}(0)=0$ for $0\le r\le\rho$, $\rho$ an integer and $Re(\sigma)>0$, $Re(\eta)>0$, has its solution given by

$$t^{\rho+\xi+\zeta-1} f(t)=\int_0^t (t-x)^{\rho-k-\sigma-1} V(t-x) g^{\rho}(x)dx \tag{3.4}$$

Where $Re(\rho-k-\sigma)>0$ and

$$\chi(x)=\frac{1}{f(r)}\sum_{\lambda=0}^{\infty}\frac{H_\lambda x^\lambda}{\Gamma(\rho-k+\lambda-\sigma)} \tag{3.5}$$

The $H_\lambda$ determined by (2.15), $h_\nu$ being given by (2.9).

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