Journal of Computer and Mathematical Sciences, *Vol.10(7),1518-1528 July 2019* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Numerical Solution of Fuzzy Differential Equations by Seventh Order Runge-Kutta Method

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(Received on: July 4, 2019)

ABSTRACT

In this paper to propose a method of computing approximation of the solution for fuzzy differential equation with initial conditions using Runge-Kutta of order seven in order to increase the order of the accuracy of the solution. This method is discussed in details followed by a complete error analysis.

Keywords: Fuzzy Differential Equations with initial value - Seventh Order Runge-Kutta Method – Fuzzy solutions – Truncation error comparisons.

1. INTRODUCTION

The topic of fuzzy differential equation has been rapidly growing in recent years. The concept of fuzzy derivatives was first introduced by S.L. Chang and L.A. Zadeh[7], it was followed up by D. Dubois and Prade[8] who used the extension principle in their approach. Puri and D.A. Ralesec[16] and R. Goetschel and W. Voxman[10] contributed towards the differential of fuzzy functions.

The fuzzy differential equation and initial value problems were extensively studied by O. Kaleva[11,12] and by S. Seikkala[17]. Numerical solution of fuzzy differential equations has been introduced by M. Ma, M. Friedman, A. Kandel[14] through Euler method and by S. Abbasbandy and T. Allahviranloo[2] by Taylor method. Runge-Kutta methods have also been studied by authors[3,15].

In this paper organized as follows: In Section 2, some basic definitions and results on fuzzy numbers and fuzzy derivatives. Section 3 contains the definition of fuzzy Cauchy problem with initial conditions. Section 4, discussed about seventh order Runge-Kutta method and defined to solve the fuzzy differential equation with initial value problem. The proposed method is illustrated and solved the numerical example in section 5, also the result is compared with Euler's method and Runge-Kutta fourth order method with the approximation solution by Runge-Kutta seventh order method.

2. PRELIMINARIES

Consider the initial value problem

$$y'(t) = \begin{cases} f(t, y(t)), & t_0 \le t \le b \\ y(t_0) = y_0 \end{cases} \tag{2.1}$$

We assume that

1. $f(t, y(t))$ is defined & continuous in the strip $t_0 \le t \le b$, $-\infty < y < \infty$ with t_0 & b are finite.
2. There exists a constant L such that for any t in $[t_0, b]$ and any two numbers y and y^*

$$\left| f(t, y) - f(t, y^*) \right| \le L \left| y - y^* \right|$$

These conditions are sufficient to prove that ∃ on $[t_0, b]$ a unique continuous differentiable function y(t) satisfying (2.1). The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i$$; where w_i are constant i and $k_i = hf(t_n + a_i h,\ y_n + \sum_{j=1}^{i-1} c_{ij} k_j)$

Most efforts to increase the order of accuracy of the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required[6]. The method proposed by Goeken. D and Johnson. O[9] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta k_i terms (i > 1) to obtain a higher order of accuracy without a corresponding increase in evaluations of f, but with the addition of evaluations of f^1.

Consider

$y(t_{n+1}) = y(t_n) + w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4 + w_5k_5 + w_6k_6 + w_7k_7 + w_8k_8 + w_9k_9 + w_{10}k_{10} + w_{11}k_{11}$

where

$k_1 = hf(t_n, y(t_n))$

$k_2 = h f(t_n+c_2h, y(t_n)+a_{21}k_1)$

$k_3 = h f(t_n+c_3h, y(t_n)+a_{31}k_1+a_{32}k_2)$

$k_4 = h f(t_n+c_4h, y(t_n)+a_{41}k_1+a_{42}k_2+a_{43}k_3)$

$k_5 = h f(t_n+c_5h, y(t_n)+a_{51}k_1+a_{52}k_2+a_{53}k_3+a_{54}k_4)$

$k_6 = h f(t_n+c_6h, y(t_n)+a_{61}k_1+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5)$

$k_7 = h f(t_n+c_7h, y(t_n)+a_{71}k_1+a_{72}k_2+a_{73}k_3+a_{74}k_4+a_{75}k_5+a_{76}k_6)$

$k_8 = h f(t_n+c_8h, y(t_n)+a_{81}k_1+a_{82}k_2+a_{83}k_3+a_{84}k_4+a_{85}k_5+a_{86}k_6+a_{87}k_7)$

$k_9 = h f(t_n+c_9h, y(t_n)+a_{91}k_1+a_{92}k_2+a_{93}k_3+a_{94}k_4+a_{95}k_5+a_{96}k_6+a_{97}k_7+a_{98}k_8)$

$k_{10} = h\, f(t_n+c_{10}h,\ y(t_n)+a_{101}k_1+a_{102}k_2+a_{103}k_3+a_{104}k_4+a_{105}k_5+a_{106}k_6+a_{107}k_7+a_{108}k_8+a_{109}k_9)$

$k_{11} = hf(t_n+c_{11}h,\ y(t_n)+a_{111}k_1+a_{112}k_2+a_{113}k_3+a_{114}k_4+a_{115}k_5+a_{116}k_6+a_{117}k_7+a_{118}k_8+a_{119}k_9+a_{1110}k_{10})$ (2.2)

The Taylor's series expansion techniques, seventh order Runge-Kutta method is given by, $y_{n+1} = y_n + \frac{41k_1 + 216k_6 + 27k_7 + 272k_8 + 27k_9 + 216k_{10} + 41k_{11}}{840}$

where

$k_1 = h\, f(t_n, y(t_n))$

$k_2 = h\, f(t_n+\frac{2h}{36},\ y(t_n)+\frac{k_1}{18})$

$k_3 = h\, f(t_n+\frac{3h}{36},\ y(t_n)+(4k_1+k_2)/60)$

$k_4 = h\, f(t_n+\frac{4h}{36},\ y(t_n)+(-181k_1+171k_2+30k_3)/180)$

$k_5 = h\, f(t_n+\frac{5h}{36},\ y(t_n)+(-902k_1+2931k_2-2040k_3+30k_4)/180)$

$k_6 = h\, f(t_n+\frac{h}{6},\ y(t_n)+(-15k_1+48k_2-31k_3+k_4+k_5)/24)$

$k_7 = h\, f(t_n+\frac{2h}{6},\ y(t_n)+(17k_1-48k_2+31k_3-k_4-k_5+12k_6)/30)$

$k_8 = h\, f(t_n+\frac{3h}{6},\ y(t_n)+(192k_1-528k_2+341k_3-11k_4-11k_5+32k_6+25k_7)/80)$

$k_9 = h\, f(t_n+\frac{4h}{6},\ y(t_n)+(54k_1-144k_2+93k_3-3k_4-3k_5+32k_6-17k_7+32k_8)/66)$

$k_{10} = h\, f(t_n+\frac{5h}{6},\ y(t_n)+(-22876k_1+64464k_2-41633k_3+1343k_4+1343k_5-656k_6-460k_7-40k_8+1815k_9)/3960)$

$k_{11} = h\, f(t_n+h,\ y(t_n)+(16139k_1-45120k_2-29140k_3-940k_4-940k_5+1828k_6-769k_7+2752k_8-1980k_9+792k_{10})/902)$ (2.3)

Definition – 2.1

A fuzzy number u as a fuzzy subset of R, u : R → [0, 1] satisfying the conditions

i). u is normal, ie $\exists\, x_0 \in R \ni u(x_0) = 1$

ii) u is a convex fuzzy set, ie) $u(tx+(1-t)y) \geq \min\{u(x), u(y)\}$, $\forall t \in [0, 1]$ & $x, y \in R$

ii). u is upper semi continuous on R

iii). $\overline{\{x \in R, u(x) > 0\}}$ is compact\

The set E is the family of fuzzy numbers and arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$ has satisfies the following requirements

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1] w.r.to any 'r'.
2. $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1] w.r.to any 'r'.
3. $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$, r-level cut is $[u]_r = \{x/u(x) \ge r\}$, $0 \le r \le 1$ as a closed & bounded interval denoted by $[u]_r = [\underline{u}(r), \overline{u}(r)]$ and clearly $[u]_0 = \{x/u(x) > 0\}$ is compact.

Definition – 2.2

A triangular fuzzy number u is a fuzzy set in E that is characterized by an ordered triple $(u_l, u_c, u_r) \in R^3$ with $u_l < u_c < u_r$ such that $[u]_0 = [u_l : u_r]$ and $[u]_1 = [u_c]$.
The membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \dfrac{x-u_l}{u_c-u_l}, & u_l \le x \le u_c \\ 1 & x = u_c \\ \dfrac{u_r-x}{u_r-u_c}, & u_c \le x \le u_r \end{cases} \quad \text{and we will have} \tag{2.4}$$

i). $u > 0$ if $u_l > 0$ ii) $u \ge 0$ if $u_l \ge 0$ iii) $u < 0$ if $u_c < 0$ iv) $u \le 0$ if $u_c \le 0$
A mapping y : I(Interval) → E as called a fuzzy process and its α - level set is denoted by $[y(t)]_\alpha = [\underline{y}(t,y), \overline{y}(t,y)]$, $t \in I$, $0 < \alpha \le I$. The Seikkala derivative y(t) of a fuzzy process is defined by $[y^1(t)]_\alpha = [\underline{y}^1(t,y), \overline{y}^{-1}(t,y)]$, $t \in I$, $0 < \alpha \le I$ provided the equation defines fuzzy number as in [12]. For u, v ∈ E and $\lambda \in \Re$, the u + v and λu can be defined by $[u + v]_\alpha = [u]_\alpha + [v]_\alpha$ and $[\lambda u]_\alpha = \lambda[u]_\alpha$, where $\alpha \in [0, 1]$ and $[u]_\alpha + [v]_\alpha$ means the addition of two intervals of $\Re$ and $[u]_\alpha$ means the product between a scalar and a subset of $\Re$. Arithmetic operation of arbitrary fuzzy numbers u = $(\underline{u}(r), \overline{u}(r))$, v = $(\underline{v}(r), \overline{v}(r))$ & $\lambda \in \Re$ can be defined as

i). u + v = $(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$ ii) u - v = $(\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$

iii). λu = $(a\,\underline{u}(r), a\,\overline{u}(r))$ if $\lambda \ge 0$, = $(a\,\overline{u}(r), a\,\underline{u}(r))$ if $\lambda < 0$

3. A FUZZY CAUCHY PROBLEM

Consider the fuzzy initial value problem

$$\left.\begin{aligned} y'(t) &= f(t, y(t)),\ 0 \le t \le T \\ y(0) &= y_0 \end{aligned}\right\} \tag{3.1}$$

where f is a continuous mapping from $R_+ \times R$ → R and $y_0 \in E$ with r-level sets $[y_0]_r = [\underline{y}(0:r), \overline{y}(0:r)]$, $r \in [0, 1]$. The extension principle of Zadeh leads to the following definition of f(t, y) then y = y(t) is a fuzzy number. $f(t, y)(s)=\sup\{y(\tau)\backslash s=f(t, \tau)\}$, $s \in R$,

$\Rightarrow [f(t,y)]_r = [\underline{f}(t,y:r), \overline{f}(t,y:r)]$, r∈[0, 1]

$\Rightarrow \underline{f}(t,y:r) = \min\{f(t,u) \setminus u \in [\underline{y}(r), \overline{y}(r)]\}$ & $\overline{f}(t,y:r) = \max\{f(t,u) \setminus u \in [\underline{y}(r), \overline{y}(r)]\}$ (3.2)

Theorem:

Let f satisfy $\left| f(t,v) - f(t,\overline{v}) \right| \le g(t, |v - \overline{v}|)$, t ≥ 0 and v, $\overline{v}$ ∈ R, (3.3)

where g : $R_+ \times R_+ \rightarrow R_+$ is a continuous mapping such that r → g(t, r) is non-decreasing an initial value problem $u^1(t) = g(t, u(t))$, $u(0) = u_0$ ... (3.3) has a solution on R_+ or $u_0 > 0$ and that $u(t) \equiv 0$ is the only solution of (3.3) for $u_0 = 0$ then the fuzzy initial value problem (3.1) has a unique fuzzy solution.

Proof : See[17].

4. SEVENTH ORDER RUNGE–KUTTA METHOD

Let the exact solution of the given equation $[y(t)]_r$ = $[\underline{y}(t:r), \underline{y}(t:r)]$ is approximated by some $[y(t)]_r$ = $[\underline{y}(t:r), \underline{y}(t:r)]$ and we define

$\underline{y}(t_{n+1}:r) - \underline{y}(t_n:r) = \sum_{i=1}^{11} w_i \underline{k_i}$ $\quad \overline{y}(t_{n+1}:r) - \overline{y}(t_n:r) = \sum_{i=1}^{11} w_i \overline{k_i}$ where w_i's are constant

$[k_i(t, y(t,r), z(t,r))]_r = [\underline{k_i}(t, y(t,r), z(t,r)), \overline{k_i}(t, y(t,r), z(t,r))]$ where i = 1 to 11

$$\underline{k_1}(t, y(t:r), z(t:r)) = hf(t_n, \underline{y}(t_n:r), \underline{z}(t_n:r))$$

$$\overline{k_1}(t, y(t:r), z(t:r)) = hf(t_n, \overline{y}(t_n:r), \overline{z}(t_n:r))$$

$$\underline{k_2}(t, y(t:r), z(t:r)) = hf\left(t_n + \frac{2h}{36}, \underline{y}(t_n:r) + \frac{1}{18}\underline{k_1}, \underline{z}(t_n:r) + \frac{1}{18}\underline{l_1}\right)$$

$$\overline{k_2}(t, y(t:r), z(t:r)) = hf\left(t_n + \frac{2h}{36}, \overline{y}(t_n:r) + \frac{1}{18}\overline{k_1}, \overline{z}(t_n:r) + \frac{1}{18}\overline{l_1}\right)$$

$$\underline{k_3}(t, y(t:r), z(t:r)) = hf\left(t_n + \frac{3h}{36}, \underline{y}(t_n:r) + (4\underline{k_1} + \underline{k_2})/60, \underline{z}(t_n:r) + (4\underline{l_1} + \underline{l_2})/60\right)$$

$$\overline{k_3}(t, y(t:r), z(t:r)) = hf\left(t_n + \frac{3h}{36}, \overline{y}(t_n:r) + (4\overline{k_1} + \overline{k_2})/60, \overline{z}(t_n:r) + (4\overline{l_1} + \overline{l_2})/60\right)$$

$$\underline{k_4}(t, y(t:r), z(t:r)) = hf\begin{pmatrix} t_n + \frac{4h}{36}, \underline{y}(t_n:r) + (-181\underline{k_1} + 171\underline{k_2} + 30\underline{k_3})/180, \\ \underline{z}(t_n:r) + (-181\underline{l_1} + 171\underline{l_2} + 30\underline{l_3})/180 \end{pmatrix}$$

$$\overline{k_4}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{4h}{36}, \overline{y}(t_n:r) + \left(-181\overline{k_1} + 171\overline{k_2} + 30\overline{k_3}\right)/180, \\ \overline{z}(t_n:r) + \left(-181\overline{l_1} + 171\overline{l_2} + 30\overline{l_3}\right)/180 \end{array}\right)$$

$$\underline{k_5}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{5h}{36}, \underline{y}(t_n:r) + \left(-902\underline{k_1} + 931\underline{k_2} - 2040\underline{k_3} + 30\underline{k_4}\right)/180, \\ \underline{z}(t_n:r) + \left(-902\underline{l_1} + 931\underline{l_2} - 2040\underline{l_3} + 30\underline{l_4}\right)/180 \end{array}\right)$$

$$\overline{k_5}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{5h}{36}, \overline{y}(t_n:r) + \left(-902\overline{k_1} + 2931\overline{k_2} - 2940\overline{k_3} + 30\overline{k_4}\right)/180, \\ \overline{z}(t_n:r) + \left(-902\overline{l_1} + 2931\overline{l_2} - 2940\overline{l_3} + 30\overline{l_4}\right)/180 \end{array}\right)$$

$$\underline{k_6}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{h}{6}, \underline{y}(t_n:r) + \left(-15\underline{k_1} + 48\underline{k_2} - 31\underline{k_3} + \underline{k_4} + \underline{k_5}\right)/24, \\ \underline{z}(t_n:r) + \left(-15\underline{l_1} + 48\underline{l_2} - 31\underline{l_3} + \underline{l_4} + \underline{l_5}\right)/24 \end{array}\right)$$

$$\overline{k_6}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{h}{6}, \overline{y}(t_n:r) + \left(-15\overline{k_1} + 48\overline{k_2} - 31\overline{k_3} + \overline{k_4} + \overline{k_5}\right)/24, \\ \overline{z}(t_n:r) + \left(-15\overline{l_1} + 48\overline{l_2} - 31\overline{l_3} + \overline{l_4} + \overline{l_5}\right)/24 \end{array}\right)$$

$$\underline{k_7}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{2h}{6}, \underline{y}(t_n:r) + \left(17\underline{k_1} - 48\underline{k_2} + 31\underline{k_3} - \underline{k_4} - \underline{k_5} + 12\underline{k_6}\right)/30, \\ \underline{z}(t_n:r) + \left(17\underline{l_1} - 48\underline{l_2} + 31\underline{l_3} - \underline{l_4} - \underline{l_5} + 12\underline{l_6}\right)/30 \end{array}\right)$$

$$\overline{k_7}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{2h}{6}, \overline{y}(t_n:r) + \left(17\overline{k_1} - 48\overline{k_2} + 31\overline{k_3} - \overline{k_4} - \overline{k_5} + 12\overline{k_6}\right)/30, \\ \overline{z}(t_n:r) + \left(17\overline{l_1} - 48\overline{l_2} + 31\overline{l_3} - \overline{l_4} - \overline{l_5} + 12\overline{l_6}\right)/30 \end{array}\right)$$

$$\underline{k_8}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{3h}{6}, \underline{y}(t_n:r) + \left(192\underline{k_1} - 528\underline{k_2} + 341\underline{k_3} - 11\underline{k_4} - 11\underline{k_5} + 32\underline{k_6} + 32\underline{k_7}\right)/80, \\ \underline{z}(t_n:r) + \left(192\underline{l_1} - 528\underline{l_2} + 341\underline{l_3} - 11\underline{l_4} - 11\underline{l_5} + 32\underline{l_6} + 25\underline{l_7}\right)/80 \end{array}\right)$$

$$\overline{k_8}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{3h}{6}, \overline{y}(t_n:r) + \left(192\overline{k_1} - 528\overline{k_2} + 341\overline{k_3} - 11\overline{k_4} - 11\overline{k_5} + 32\overline{k_6} + 25\overline{k_7}\right)/80, \\ \overline{z}(t_n:r) + \left(192\overline{l_1} - 528\overline{l_2} + 341\overline{l_3} - 11\overline{l_4} - 11\overline{l_5} + 32\overline{l_6} + 25\overline{l_7}\right)/80 \end{array}\right)$$

$$\underline{k_9}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{4h}{6}, \underline{y}(t_n:r) + \left(54\underline{k_1} - 142\underline{k_2} + 93\underline{k_3} - 3\underline{k_4} - 3\underline{k_5} + 32\underline{k_6} - 17\underline{k_7} + 32\underline{k_8}\right)/66, \\ \underline{z}(t_n:r) + \left(54\underline{l_1} - 142\underline{l_2} + 93\underline{l_3} - 3\underline{l_4} - 3\underline{l_5} + 32\underline{l_6} - 17\underline{l_7} + 32\underline{l_8}\right)/66 \end{array}\right)$$

$$\overline{k_9}(t, y(t:r), z(t:r)) = hf\left(\begin{array}{l} t_n + \frac{4h}{6}, \overline{y}(t_n:r) + \left(54\overline{k_1} - 142\overline{k_2} + 93\overline{k_3} - 3\overline{k_4} - 3\overline{k_5} + 32\overline{k_6} - 17\overline{k_7} + 32\overline{k_8}\right)/66, \\ \overline{z}(t_n:r) + \left(54\overline{l_1} - 142\overline{l_2} + 93\overline{l_3} - 3\overline{l_4} - 3\overline{l_5} + 32\overline{l_6} - 17\overline{l_7} + 32\overline{l_8}\right)/66 \end{array}\right)$$

$$\underline{k}_{10}(t,y(t:r),z(t:r)) = hf\begin{pmatrix} t_n+\frac{5h}{6},\underline{y}(t_n:r)+\begin{pmatrix}-22876\underline{k}_1+64464\underline{k}_2-41633\underline{k}_3+1343\underline{k}_4+\\1343\underline{k}_5-656\underline{k}_6-460\underline{k}_7-40\underline{k}_8+1815\underline{k}_9\end{pmatrix}/3960,\\ \underline{z}(t_n:r)+\begin{pmatrix}-22876\underline{l}_1+64464\underline{l}_2-41633\underline{l}_3+1343\underline{l}_4+\\1343\underline{l}_5-656\underline{l}_6-460\underline{l}_7-40\underline{l}_8+1815\underline{l}_9\end{pmatrix}/3960\end{pmatrix}$$

$$\overline{k}_{10}(t,y(t:r),z(t:r)) = hf\begin{pmatrix} t_n+\frac{5h}{6},\overline{y}(t_n:r)+\begin{pmatrix}-22876\overline{k}_1+64464\overline{k}_2-41633\overline{k}_3+13433\overline{k}_4+\\1343\overline{k}_5-656\overline{k}_6-460\overline{k}_7-40\overline{k}_8+1815\overline{k}_9\end{pmatrix}/3960,\\ \overline{z}(t_n:r)+\begin{pmatrix}-22876\overline{l}_1+64464\overline{l}_2-41633\overline{l}_3+13433\overline{l}_4+\\1343\overline{l}_5-656\overline{l}_6-460\overline{l}_7-40\overline{l}_8+1815\overline{l}_9\end{pmatrix}/3960\end{pmatrix}$$

$$\underline{k}_{11}(t,y(t:r),z(t:r)) = hf\begin{pmatrix} t_n+h,\underline{y}(t_n:r)+\begin{pmatrix}16139\underline{k}_1-45120\underline{k}_2-29140\underline{k}_3-940\underline{k}_4-940\underline{k}_5+\\1028\underline{k}_6-769\underline{k}_7+2752\underline{k}_8-1980\underline{k}_9+792\underline{k}_{10}\end{pmatrix}/902,\\ \underline{z}(t_n:r)+\begin{pmatrix}16139\underline{l}_1-45120\underline{l}_2-29140\underline{l}_3-940\underline{l}_4-940\underline{l}_5+\\1028\underline{l}_6-769\underline{l}_7+2752\underline{l}_8-1980\underline{l}_9+792\underline{l}_{10}\end{pmatrix}/902\end{pmatrix}$$

$$\overline{k}_{11}(t,y(t:r),z(t:r)) = hf\begin{pmatrix} t_n+h,\overline{y}(t_n:r)+\begin{pmatrix}16139\overline{k}_1-45120\overline{k}_2-29140\overline{k}_3-940\overline{k}_4-940\overline{k}_5+\\1028\overline{k}_6-769\overline{k}_7+2752\overline{k}_8-1980\overline{k}_9+792\overline{k}_{10}\end{pmatrix}/902,\\ \overline{z}(t_n:r)+\begin{pmatrix}16139\overline{l}_1-45120\overline{l}_2-29140\overline{l}_3-940\overline{l}_4-940\overline{l}_5+\\1028\overline{l}_6-769\overline{l}_7+2752\overline{l}_8-1980\overline{l}_9+792\overline{l}_{10}\end{pmatrix}/902\end{pmatrix}$$

$$F(t,y(t:r),z(t:r)) = \frac{\begin{Bmatrix}41\underline{k}_1(t,y(t:r),z(t:r))+216\underline{k}_6(t,y(t:r),z(t:r))+27\underline{k}_7(t,y(t:r),z(t:r))+\\272\underline{k}_8(t,y(t:r),z(t:r))+27\underline{k}_9(t,y(t:r),z(t:r))+216\underline{k}_{10}(t,y(t:r),z(t:r))\end{Bmatrix}}{840}$$

$$G(t,y(t:r),z(t:r)) = \frac{\begin{Bmatrix}41\overline{k}_1(t,y(t:r),z(t:r))+216\overline{k}_6(t,y(t:r),z(t:r))+27\overline{k}_7(t,y(t:r),z(t:r))+\\272\overline{k}_8(t,y(t:r),z(t:r))+27\overline{k}_9(t,y(t:r),z(t:r))+216\overline{k}_{10}(t,y(t:r),z(t:r))\end{Bmatrix}}{840} \quad (4.1)$$

$$\underline{y}(t_{n+1}:r) = \underline{y}(t_n:r)+F\left[t_n,\underline{y}(t_n:r),\underline{z}(t_n:r)\right]$$

$$\overline{y}(t_{n+1}:r) = \overline{y}(t_n:r)+G\left[t_n,\overline{y}(t_n:r),\overline{z}(t_n:r)\right]$$

The exact and approximate solution at t_n, $0 \le n \le N$ are denoted by

$[Y(t_n)]_r = \left[\underline{Y}(t_n:r),\overline{Y}(t_n:r)\right]$

$[y(t_n)]_r = \left[\underline{y}(t_n:r),\overline{y}(t_n:r)\right]$ respectively.

Finally, the convergence conditions for the fuzzy solutions is

$$\lim_{h\to 0} \underline{y}(t_{n+1} : r) = \underline{Y}(t_{n+1} : r)$$

$$\lim_{h\to 0} \overline{y}(t_{n+1} : r) = \overline{Y}(t_{n+1} : r)$$

Lemma:

Let a sequence of numbers $\{W\}_{n=0}^{N}$ satisfy

$|W_{n+1}| \le A|W_n| + B$, $0 \le n \le N-1$ or some given positive constants A and B then

$|W_n| \le A^n |W_0| + B\dfrac{A^n - 1}{A-1}$, $0 \le n \le N-1$

Proof : See[14]

Lemma:

Let a sequence of numbers $\{W\}_{n=0}^{N}$ and $\{V\}_{n=0}^{N}$ satisfy the condition

$$|W_{n+1}| \le |W_n| + A \max\{|W_n|, |V_n|\} + B \text{ and}$$

$|V_{n+1}| \le |V_n| + A \max\{|W_n|, |V_n|\} + B$ for some given positive constants A & B and denote $U_n = |W_n| + |V_n|$, $0 \le n \le N$ where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$

Proof : See[14]

Theorem:

Let F(t, u, v) and G(t, u, v) belongs to $C^4(K)$ and let the partial derivatives of F and G be bounded over K, then for arbitrary fixed value r, $0 \le r \le 1$ are approximate solutions converge to the exact solutions of $\underline{Y}(t_n : r)$ and $\overline{Y}(t_n : r)$ uniformly in t.

Proof : See[14]

5. NUMERICAL EXAMPLE

Consider the fuzzy initial value problem,

$y^1(t) = \dfrac{y}{1+x^2}$, $t \in [0, 1]$ with $y(0) = (0.8 + 0.2r, 1.25-0.25r)$ where $0 \le r \le 1$

Solution:

The derived solutions of the given fuzzy initial value problem is

$$\underline{y}(t : r) = \underline{y}(t : r)e^{\tan^{-1}(t)} \quad \text{and} \quad \overline{y}(t : r) = \overline{y}(t : r)e^{\tan^{-1}(t)}$$

then at t = 1, The solution is $y(1 : r) = [(0.8 + 0.2 r) e^{\arctan(1)}, (1.25 - 0.25 r) e^{\arctan(1)}]$, $0 \le r \le 1$

Table 5.1
Exact and Approximate Solutions by RK order seven

r	Exact Solution		Runge-Kutta 7th order Method h=0.1	
	<u>Y</u>	**}Y**	**<u>y</u>**	**}y**
0.0	1.75462404059041	2.74160006342252	1.75550210475921	2.20550227165222
0.1	1.79848964160517	2.68676806215407	1.79939341545104	2.20439291000366
0.2	1.84235524261993	2.63193606088562	1.84327852725982	2.20327901840209
0.3	1.88622084363469	2.57710405961717	1.88716769218444	2.20216822624206
0.4	1.93008644464945	2.52227205834872	1.93105638027191	2.20105671882629
0.5	1.97395204566421	2.46744005708027	1.97494149208068	2.19994544982910
0.6	2.01781764667897	2.41260805581182	2.01882886886596	2.19883012771606
0.7	2.06168324769373	2.35777605454337	2.06271672248840	2.19771814346313
0.8	2.10554884870849	2.30294405327492	2.10660243034362	2.19660425186157
0.9	2.14941444972325	2.24811205200647	2.15048909187316	2.19548773765563
1.0	2.19328005073802	2.19328005073802	2.19438052177429	2.19438052177429

Table 5.2
Approximate Solutions by RK order four and Euler's Method

r	Runge-Kutta 4th order method h=0.1		Euer's Method h=0.1	
	<u>Y</u>	**}Y**	**<u>y</u>**	**}y**
0.0	1.8401323557	2.2901322842	1.7141865492	2.7426986694
0.1	1.8861352205	2.2911350727	1.7713263035	2.6969873905
0.2	1.9321382046	2.2921388149	1.8284659386	2.6512756348
0.3	1.9781421423	2.2931420803	1.8856054544	2.6055636406
0.4	2.0241453648	2.2941451073	1.9427449703	2.5598516464
0.5	2.0701482296	2.2951481342	1.9998844862	2.5141406059
0.6	2.1161515713	2.2961518764	2.0570237637	2.4684288502
0.7	2.1621553898	2.2971553802	2.1141636372	2.4227168560
0.8	2.2081587315	2.2981584072	2.1713032722	2.3770053387
0.9	2.2541615963	2.2991616726	2.2284424305	2.3312940598
1.0	2.3001649380	2.3001649380	2.2855825424	2.2855825424

Now, comparing the approximate solutions by using the seventh order fuzzy Runge-Kutta method, Runge-Kutta fouth order method also Euler's method and the exact solutions, we get the following list of truncated errors.

Table 5.3
Error Analysis between RK order seven, RK order four and Euler's method

r	Error Comparison between Exact and Approximate Solution with h=0.1					
	Runge Kutta 7th order Method		Runge Kutta 4th order Method		Euler's Method	
	<u>Y</u>	**}Y**	**<u>Y</u>**	**y**	**<u>y</u>**	**y**
0.0	0.00087806416880	0.53609779177030	0.085508315109588	0.451467779222519	0.04043749139041	0.00109860597748
0.1	0.00090377384587	0.48237515215041	0.087645578894827	0.395632989454069	0.02716333810517	0.01021932834593
0.2	0.00092328463989	0.42865704248353	0.089782961980067	0.339797245985618	0.01388930401993	0.01933957391438
0.3	0.00094684854975	0.37493583337511	0.091921298665307	0.283961979317168	0.00061538923469	0.02845958098283
0.4	0.00096993562246	0.32121533952243	0.094058920150546	0.228126951048718	0.01265852565055	0.03757958805128
0.5	0.00098944641647	0.26749460725117	0.096196183935786	0.172291922880267	0.02593244053579	0.04670054881973
0.6	0.00101122218699	0.21377792809576	0.098333924621026	0.116456179411817	0.03920611702103	0.05582079438818
0.7	0.00103347479467	0.16005791108024	0.100472142106266	0.060620674343366	0.05248038950627	0.06494080145663
0.8	0.00105358163513	0.10633980141335	0.102609882791505	0.004785646074916	0.06575442349151	0.07406128542508
0.9	0.00107464214991	0.05262431435084	0.104747146576745	0.051049620593535	0.07902798077675	0.08318200779353
1.0	0.00110047103627	0.00110047103627	0.106884887261985	0.106884887261985	0.09230249166198	0.09230249166198

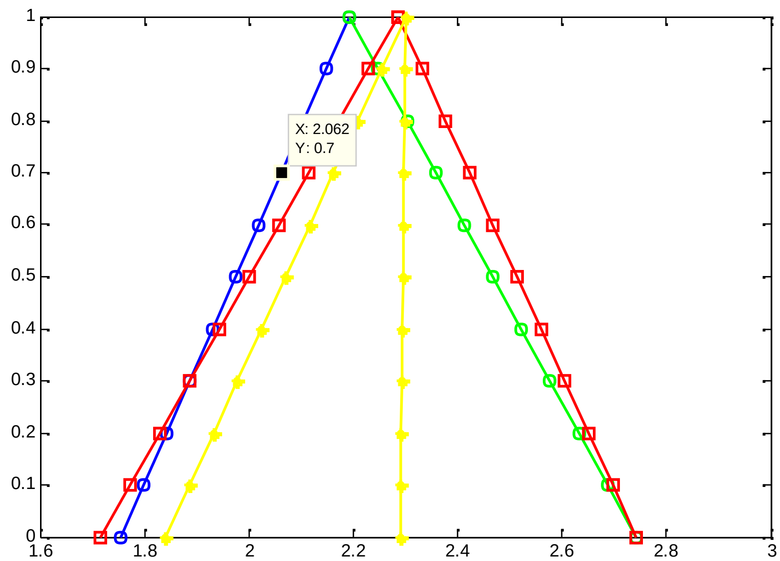

Figure 5.1
Exact and Approximate Solutions by RK order Seven and Euler's method

6. CONCLUSION

In this work, we have used the proposed seventh order Runge-Kutta method to find the numerical solution of fuzzy differential equations. Taking into account the convergence order of the Euler method is O(h), a higher order of convergence O(h^3) is obtained by the proposed method and by the method proposed in[15]. Comparison of the solutions of example 5 shows that the proposed method gives a better solution than the Euler method and by the Runge-Kutta fourth order method.

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