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The Non-Negative Solutions of Difference Equation with Time Delay

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ABSTRACT

The Non-Negativity solutions are obtained for the difference equation with time delay $\Delta u_n = F(u_n) + G(u_{n-\sigma})$ and $\Delta u_n = a u_{n-\sigma}\left(1 - \frac{u_{n-\sigma}}{K}\right) \; for \;\; n > 0$. The method uses techniques based on Lipchitz functions. Example is inserted to illustrate the result.

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INTRODUCTION

Difference equations provide an important frame work for analysis of dynamical phenomena in biology, ecology, economics and so forth and the various mathematical models of biological systems in terms of delay difference equation has been studied in different context[1-7]. For example population dynamics discrete systems adequately describe organisms for which births occur in regular usually short, breeding seasons.

Recently there has been a lot of interest in non-negativity of the solution of population dynamics model of difference equation. Motivated by the reference[1-13], in this paper, we considered the non-negativity of solutions of Delay Difference Equations of the form.

$$\Delta u_n = F(u_n) + G(u_{n-\sigma}) \quad (1.1)$$

Where Δ the forward difference operator is defined by $\Delta u_n = u_{n+1} - u_n$ and $\{u_n\} \in \mathbb{R}^n, n \in \mathbb{N}$, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipchitz continuous function and $G : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function,

$$F = (F_{1,\ldots\ldots} F_n), G = (\mathrm{G}_{1,\ldots\ldots} G_n)$$

and constant $\sigma > 0$. Let $\Phi : [-\sigma, 0] \to \mathbb{R}^n$ be continuous function and

$$u_n = \Phi(n) \quad for \;\; n \in [-\tau, 0] \quad (1.2)$$

be an initial data of equation (1.1).

Consider the Equation

$$\Delta u_n = F(u_n) \quad for \; n \geq 0$$
$$u_o = \Phi(0) \quad (1.3)$$

Theorem 1.1

Assume that equation (1.3), for each $\Phi(0) \geq 0$, has non-negativity solution and the following inequality holds

1) $G_i(s) \geq 0, \; \forall s \in (\mathbb{R}^+)^n, \; i = 1, \ldots\ldots, \mathrm{n}$ (1.4)

Then the solution of equation (1.1), (1.2) with $\Phi(n) = 0 \;\; for \;\; n \in [-\sigma, 0]$ is non-negative on the interval on which it exists.

2) $G_i(\hat{s}) < -(F_i(0) + \in), \;\; \forall \hat{s} \in (\mathbb{R}^+)^n, \; i = 1, \ldots\ldots, \mathrm{n}$ (1.5)

Then there exists a function $\Phi(0) \geq 0$, such that the corresponding solution of (1.1), (1.2) becomes negative in a finite interval time.

Proof of (1)

Let the sequence $\{u_n\}$ be solution of the equation (1.1), (1.2). Let $n \in [0, \sigma]$ then we have $n - \sigma \in [-\sigma, 0]$ inequality (1.4) implies

$$\Delta u_{i_n} < F_i(u_n) \quad \forall i \in \{1, \ldots. n\}, \; \forall n \geq 0 \quad (1.6)$$

Assume that sequence $\{u_{i_n}\}$ can be negative for some $i = \{1, \ldots . \mathrm{n}\}$. Then there exists $n_0 \geq 0$ such that

$$u_{n_o} = 0 \quad and \quad \Delta u_{n_o} \leq 0 \quad (1.7)$$

I. But the autonomous equation $\Delta x_n = F(x_n)$ has non-negative solution and solving this equation with initial-condition (u_0, u_{n_o}) , we obtain $F_i(u_{n_o}) \geq 0. \;\; for \; G(u_{n_o}) > 0.$ We have $\Delta u_{i_{n_0}} > F_i(u_0) \geq 0,$ Which is contradicts (1.7).

It is easy to proof for $G(u_0) = 0$, we also $u_n \geq 0$. By induction on k , the inequality $u_n \geq 0$ holds for all k and for $n \in [k\sigma, (k+1)\sigma]$. Hence

$$u_{in} \geq 0, \; \forall n \geq 0, \; for \;\; i = 1, \ldots n \quad (1.8)$$

This complete the proof of (1)

Proof of (2)

Let $\hat{s} = (\hat{s}_1, \ldots . \hat{s}_n)$ and let $\Phi = (\Phi_1, \ldots .. \Phi_n)$ be given:

$$\Phi_j(n)=\left\{\begin{array}{ll}\hat{s}_j & for\ n\in\left[-\sigma,-\frac{\sigma}{2}\right] \\ -\frac{2\hat{s}_j}{\tau}n & for\ n\in\left[-\frac{\sigma}{2},0\right]\end{array}\right\}\ for\ j=1,\ldots n \quad (1.9)$$

Therefore

$$u_{i_0}=0 \quad and \quad \Delta u_{i_0}=F_i(0)+G_i(\hat{s})<0 \quad (1.10)$$

Above shows that there exists $n>0$ $such\,that$ $u_{i0}<0.$

Example:1

Let $\{u_n\}$ be a solutions of the equation (1.1), (1.2), with $\sigma=0$ be non-negative for each non-negative initial condition. Then the Equation (1.1) becomes

$$\Delta u_n=u_n-\frac{u_{n-\sigma}}{2} \quad (1.11)$$

All the solutions of equation (1.11) with nonnegative initial condition have nonnegative solutions $u_n=\left(\frac{3}{2}\right)^n$ and negatives values with $\sigma=0$ (See Fig:1)

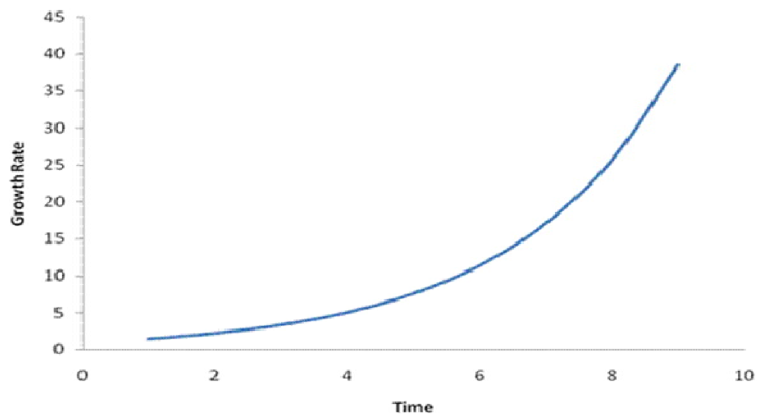

Fig.1 Time Vs Growth Rate

The Non-negativity of solutions of the Logistic Difference Equation with Time Delay

We show that the solutions of the logistics equation with time delay can become negative in an finite interval of time. Consider the equation

$$\Delta u_n=au_{n-\sigma}\left(1-\frac{u_{n-\sigma}}{K}\right) \quad for \quad n>0$$
$$u_n=\phi(n) \qquad for\ n\in[-1,0] \quad (2.1)$$

Where a is the growth rate, K the environment capacity and τ is positive constants. The scale variable x and time, we obtain

$$\Delta u_n=a\sigma u_{n-1}\left(1-u_{n-1}\right) \quad for \quad n>0$$
$$u_n=\phi(n) \qquad for\ n\in[-1,0] \quad (2.2)$$

Since $K=1$ $\ and\ initial\ condition$ $0\le\phi(n)\le1.$ The solution of equation (2.2) can have negative values.

Consider the following Equations

$$W_1 u_n = 1 + \frac{1}{4}u_n - \frac{1}{8}(u_n)^2 - \frac{1}{48}(u_n)^3$$

and (2.3)

$$W_2 u_n = 1 - \frac{1}{4}(u_n)^2 - \frac{1}{16}(u_n)^3 . \quad (2.4)$$

Let p_1 and p_2 be the greatest roots of W_1 *and* W_2 respectively, i.e.,

$$p_1 = \max\{u_n \in R : W_1(u_n) = 0\},$$

$$p_2 = \max\{u_n \in R : W_2(u_n) = 0\}.. \quad (2.5)$$

Theorem 2.1

Assume $0 \le \phi(n) \le 1, \quad for\ n \in [-1, 0]$ (2.6) then

(i) If $a\sigma > p_{1,}$ then there exists a function $\phi(n),$ which satisfies condition (2.6) and the corresponding solution to the equation (2.2) has negative values;

(ii) If $a\sigma < p_{2,}$ then there exists a function $\phi(n),$ then the solution to the equation (2.2) has non-negative values

Proof of the (i)

Let

$$\phi(n) = \begin{cases} \frac{1}{2} & for\ n \in [-1, 0] \\ 1 & for\ n = 0 \end{cases} \quad (2.7)$$

and $\{u_n\}$ be a solution of the equation (2.2) on the interval $[n-1, n]$. It is easy to see that

$$u_{1n} = 1 + \sum_{s=0}^{n} \frac{a\sigma}{4} = 1 + \frac{a\sigma}{4}n.. \quad (2.9)$$

The proof is similar to the case $\{u_{2n}\}$ Consequently we have

$$u_{2(2)} = 1 + \frac{a\sigma}{4} - \frac{(a\sigma)^2}{8} - \frac{(a\sigma)^3}{48} = W_1(a\sigma) . \quad (2.11)$$

Thus $a\sigma > p_{1,}$, then $W_1(a\sigma) = u_{2(2)} < 0$

Proof of the (ii)

To show that if condition (2.6) is fulfilled then the following inequality holds:

$$u_n \le 1 + \frac{a\sigma}{4} \qquad \forall n > 0 \quad (2.12)$$

If $u_n > 1$, then there must exist a point $n_0 < n\ such\ that\ u_{n_0} = 1.$ we have

$$u_n \le 1 + \sum_{s=n_0-1}^{n_o} \frac{a\sigma}{4} = 1 + \frac{a\sigma}{4} \quad (2.14)$$

On the other hand, we have

$$u_n \ge 1 + \sum_{s=n_0-1}^{n_o} \left(1 + \frac{a\sigma}{4}\right)\left(-\frac{a\sigma}{4}\right) = W_2(a\sigma). \quad (2.15)$$

Thus $a\sigma < p_{2,}$, then $W_2(a\sigma) > 0 \quad and \quad x(n) \ge 0$

CONCLUSION

Theorem 1.2 shows that there exist a initial condition $\Phi(n)$, such that the solution of the education (1.1) (1.2) becomes negative. But the non-negativity delay is introduced to the equation. The Logistics delayed equation of the form (2.1) then the solution is negative for non-negative initial condition and behavior of its solutions, for small values of $a\sigma$

Example 3.1

Consider the difference equation

$$\Delta u_n = au_{n-\sigma}\left(1-\frac{u_{n-\sigma}}{K}\right) \quad for \quad n>0\,. \tag{3.1}$$

Here $\sigma=-1, \quad K=1, \quad a=1$.

All conditions of Theorem (2.1) are satisfied. Hence all solutions of equation (3.1) are non-negative.

In fact $\{u_n\}=2^{1-2^{-n}}$ is one such a solution of equation (3.1) (See FIG:2)

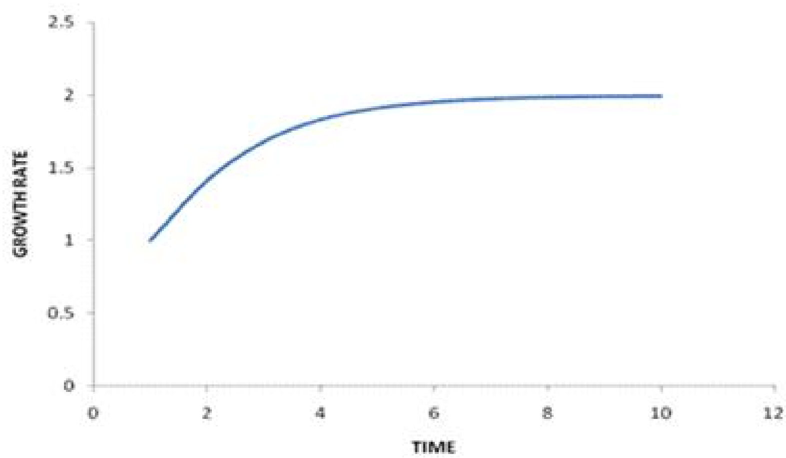

Fig.2 Time Vs Growth Rate

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