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Total Outer Independent Monophonic Number of a Graph

P. Arul Paul Sudhahar[1] and A. J. Bertilla Jaushal[2]

[1]Department of Mathematics,
Rani Anna Govt. College (W), Tirunelveli – 627 008, Tamil Nadu, INDIA.
[2]Department of Mathematics,
Nanjil Catholic College of Arts and Science,
Kaliyakkavilai – 629 153, Kanyakumari District, Tamil Nadu, INDIA.
Manonmaniam Sundaranar University, Tirunelveli -627 012, Tamil Nadu, INDIA.

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ABSTRACT

We initiate the study of total outer independent monophonic in graphs. A set of vertices M of a graph G is called a total monophonic set if M is a monophonic set and its induced subgraph has no isolated vertices. The minimum cardinality of all total monophonic sets of M is called the total monophonic number and is denoted by $m_t(G)$. It is shown that For every pair a, b of integers with $2 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $m(G) = a$ and $mt^{oi}(G) = b$.

AMS Subject classification: 05C12.

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1. INTRODUCTION

Let $G = (V, E)$ be a graph and n be the number of vertices and m be the number of edges. Thus the cardinality of $V(G) = m$ and the cardinality of $E(G) = n$. We consider a finite undirected graph without loops or multiple edges. For the basic graph theoretic notations and terminology we refer to Buckley and Harary. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G . A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic.

The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighbourhood. A vertex v is an extreme vertex if the subgraph induced by its neighbourhood is complete. A vertex v in a connected graph G is a cut vertex of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be a semi-extreme vertex if $\Delta(< N(v) >) = |N(v)| - 1$. A graph G is said to be semi-extreme graph if every vertex of G is a semi-extreme vertex. An acyclic graph is called a tree .

A subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. A *monophonic set* of G is a set $M \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The *monophonic number* $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* of G. A subset of $V(G)$ is independent if there is no edge between any two vertices of that set. A total monophonic set of a graph G is a monophonic set M such that the subgraph induced by M has no isolated vertex. The minimum cardinality of a total monophonic set of G is its total monophonic number and is denoted by $mt(G)$. A total monophonic set of size $mt(G)$ is said to be a mt- set.

2. TOTAL OUTER INDEPENDENT MONOPHONIC NUMBER OF A GRAPH

Definition 2.1: A monophonic set $S \subseteq V$ is said to be total outer independent monophonic set, abbreviated TOIMS if it is a total monophonic set and $\langle V - S \rangle$ is independent.The minimum cardinality of a total outer independent monophonic set, denoted by $mt^{oi}(G)$ is called the total outer independent monophonic number of G.

Example 2.2: For the graph given in Fig.2.1, it is clear that $M_1 = \{v_1, v_2, v_7\}$ is the monophonic set of G so that $m(G) = 3$. It is verified that the set $M_2 = \{v_1, v_2, v_6, v_7\}$ is the minimum total monophonic set so that $mt(G) = 4$. Also, the set $M_3 = \{v_1, v_2, v_4, v_6, v_7\}$ is the outer independent total monophonic set and so $mt^{oi}(G) = 5$.

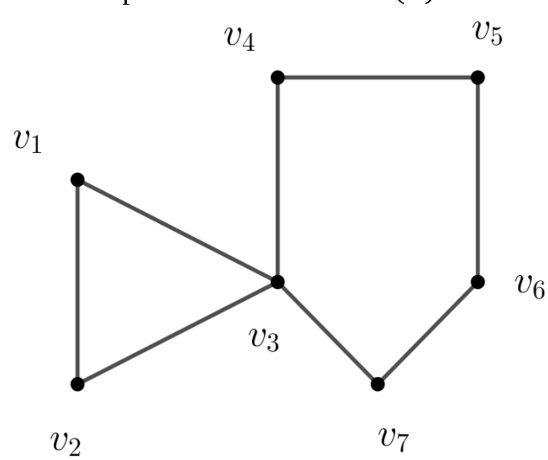

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Fig 2.1

3. RESULTS

Observation 3.1. For every graph G, we have $mt^{oi}(G) \geq mt(G)$.
Observation 3.2. Every support vertex of a graph G is in every TOIMS of G.
Observation 3.3. If $p \geq 3$ is an integer, then $mt^{oi}(C_p) = \left\lceil \frac{p+2}{2} \right\rceil$.
Observation 3.4. For every integer $n \geq 3$, we have $mt^{oi}(P_n) = \left\lceil \frac{n+3}{2} \right\rceil$.
Observation. 3.5. For every integer $n \geq 4$, $mt^{oi}(W_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$
Observation 3.6. If p and q are positive integers, then $mt^{oi}(K_{p,q}) = \min\{p, q\} + 1$.

Theorem 3.7. For any connected graph G of order p, $2 \leq mt(G) \leq mt^{oi}(G) \leq p$.
Proof. A total monophonic set needs atleast two vertices and therefore $mt\,(G) \geq 2$. Clearly the set of all vertices of G is an TOIM set of G so that $mt^{oi}(G) \leq p$.

Theorem 3.8. For the complete graph $K_p, p \geq 2$, $mt^{oi}(K_p) = p$.
Proof. Since every vertex of the complete graph $K_p, p \geq 2$ is an extreme vertex , the vertex set of K_p is the unique TOIM set of K_p. Thus $mt^{oi}(K_p) = p$.

Theorem 3.9. Let G be a connected graph of order $p \geq 2$, then $mt^{oi}(G) = p$ if and only if G is the complete graph on p vertices.
Proof. Suppose $G = K_p$.Then by theorem 3.8, $mt^{oi}(K_p) = p$. Conversely let $mt^{oi}(K_p) = p$. Suppose that G is not a complete graph, then $mt^{oi}(G) \leq p - 1$, which is a contradiction.
Remark 3.10. The bounds in theorem 3.7 are sharp. For the complete graph $K_p, p \geq 2$, $mt^{oi}(K_p) = p$. For the graph given in Fig 2.1, $p = 7, mt(G) = 4, mt^{oi}(G) = 5$. Thus the theorem holds.

Theorem 3.11. For any connected graph G with vertex count p, $mt^{oi}(G) > p - mt(G)$.
Proof. Let M be a minimum TOIM set of G. Then it follows that $mt(G) > |V - M| > |V| - |M| > p - mt^{oi}(G)$.
Remark 3.12. The bounds in theorem 3.11 are sharp. For the graph G given in Fig.2.2, $(G) = 4, p = 7, mt^{oi}(G) = 5$.

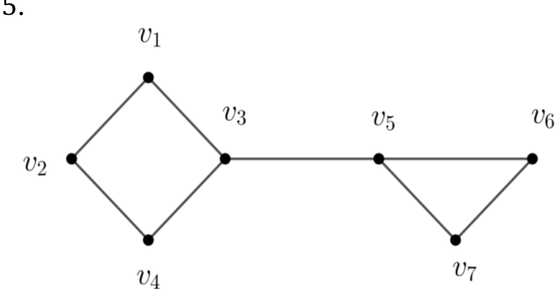

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Fig 2.2

Theorem 3.13. Let G be a graph. We have
$mt^{oi}(G) = 4 \ iff \ G \in \{P_4, P_5\}$
$mt^{oi}(G) = n \ iff \ G \in P_4$
Proof. Obviously, $mt^{oi}(P_4) = 4 = n$ and $mt^{oi}(P_5) = 4$.
Assume that for some graph G, we have $mt^{oi}(G) = 4$. Let M be a $mt^{oi}(G)$ - set. If all vertices of G belong to the set M, then the graph G has 4 vertices. Consequently, $G = P_4$. Now let x be a vertex of $V(G) - M$.Since the set $V(G) - M$ is independent, the vertex x cannot have a neighbour in G.This implies that G is a path P_5.
Now, assume that for some graph G, we have $mt^{oi}(G) = n$.If G has atleast 5 vertices, then it has a vertex, say x of degree atleast 2. Let us observe that $M - \{x\}$ is a independent set of the graph G. This implies that $mt^{oi}(G) \leq n - 1$. Therefore, the graph G has exactly 4 vertices and consequently, it is a path P_4.
Proposition 3.14. Let G be a graph. For every vertex v of G, we have $mt^{oi}(G) - 1 \leq mt^{oi}(G - v) \leq mt^{oi}(G) + d_G(v) - 1$.
Proof. Let M be a $mt^{oi}(G)$-set. If $\notin M$, then observe that M is a TOIMS of the graph $G - v$. Now assume that $v \in M$. Let us observe that $M \cup N_G(v)\backslash\{v\}$ is a TOIMS of the graph $G - v$. Therefore, $mt^{oi}(G - v) \leq |M \cup N_G(v)\backslash\{v\}| \leq |M\{v\}| + |N_G(v)| = mt^{oi}(G) + d_G(v) - 1$. Now, let M' be any $mt^{oi}(G - v)$ set. It is easy to see that $M' \cup \{v\}$ is a TOIMS of the graph G.Thus $mt^{oi}(G) \leq mt^{oi}(G - v) + 1$.

Theorem 3.13. For every pair a, b of integers with $2 \leq a \leq b$ and $b \geq 2$, there exists a connected graph G such that $m(G) = a$ and $mt^{oi}(G) = b$.
Proof. Case 1. If $a = b$. Let $G = K_2$, then $m(G) = 2$ and $mt^{oi}(G) = 2$.
Case 2. If $a = 2, b = 3$. Then for the graph given in Fig 2.3**,** $\{v, x\}$ is the minimum monophonic set of G, so that $m(G) = 2$ and $\{v, x, y\}$ is the total outer independent monophonic set of G, so that $mt^{oi}(G) = 3$.

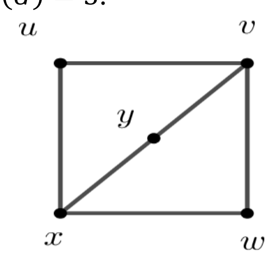

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Fig 2.3

Case 3. If $a = 2, b > 3$. Let $C_4: u, v, w, x, u$ be a cycle of order 4.Let G be the graph obtained from Fig. 2.4. By adding $b - 2$ new vertices $y_1, y_2, \dots, y_{b-2}$ and joining each $y_i (1 \leq i \leq b - 2)$ with x, v and w. Also, each y_i is adjacent to every y_i. It is clear that $M_1 = \{v, x\}$ is a minimum monophonic set of G, so that $m(G) = 2 = a$. Also, $M_2 = M_1 \cup \{w\}$ is the minimum total monophonic set of G, and so $V(G) - M_2$ is not independent. Therefore, $M_3 = M_2 \cup$

$\{y_1, y_2, \dots, y_{b-3}\}$ is the total outer independent monophonic set of G, so that $mt^{oi}(G) = 3 + b - 3 = b$.

Case 4. If $a \geq 3, b \geq 4, b \neq a + 3$. Let G be the graph given in Fig 2.5, obtained from the path on 3 vertices $P_3: z, u, w$ by adding $a - 1$ new vertices $u_1, u_2, \dots, u_{a-1}$ to z and joining each $v_i (1 \leq i \leq b - a - 1)$ with z, u and w. Also, each v_i is adjacent to every v_i.

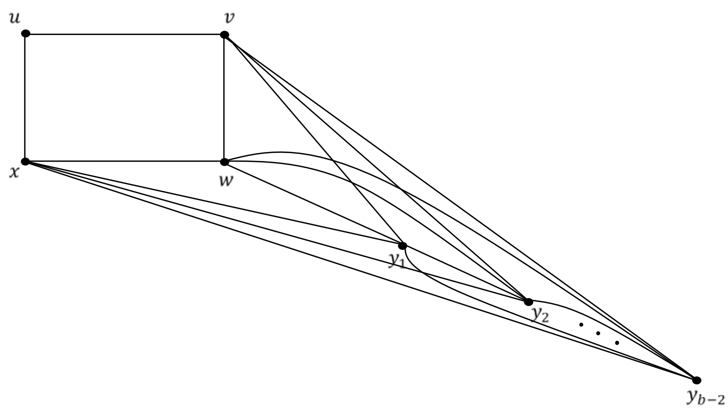

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Fig 2.4

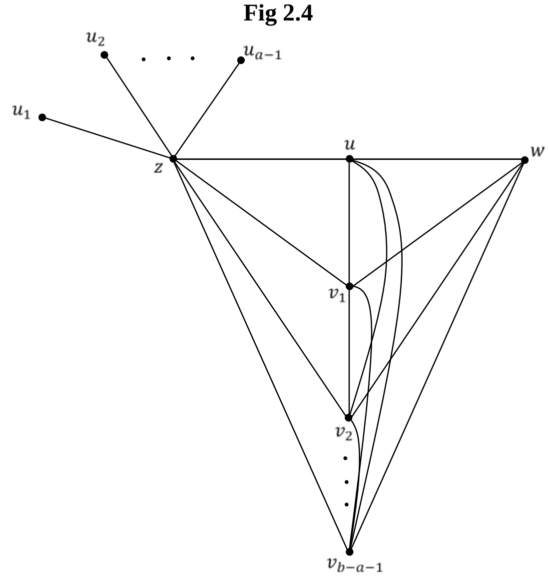

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Fig 2.5

Let $M_1 = \{u_1, u_2, \dots, u_{a-1}, u\}$ be the extreme vertices of G. It is clear that M_1 is the monophonic set of G so that $m(G) = a$. Also, $M_2 = M_1 \cup \{z, v\}$ is a total monophonic set of G. Therefore, $V(G) - M_2$ is not an independent set. Now, it is clear that $M_3 = M_2 \cup \{v_1, v_2, \dots, v_{b-a-2}\}$ is the TOIM set of G so that $mt^{oi}(G) = a + 2 + b - a - 2 = b$.

Case 5. If $a \geq 3, b \geq 4, b = a + 3$. Let G be the graph given in Fig.2.6**,** Obtained from the path on 5 vertices $P_5: u_1, u_2, u_3, u_4, u_5$ by adding $b - 2$ new vertices $v_1, v_2, \dots, v_{b-a}, w_1, w_2, \dots, w_{a-2}$ to P_5 and joining each $v_i (1 \leq i \leq b - a)$ with u_2, u_4 and joining each $w_i (1 \leq i \leq a - 2)$ with u_3. Let $M_1 = \{w_1, w_2, \dots, w_{a-2}, u_1, u_5\}$ be the set of all end vertices of G. It is clear that M_1 is the unique minimum monophonic set of G so that $m(G) = a$. Let $M_2 = M_1 \cup \{u_2, u_3, u_4\}$. It is clear that M_2 is the total outer independent monophonic set of G and $V(G) - M_2$ is independent. Therefore, $M_2 = a + 3 = b$.

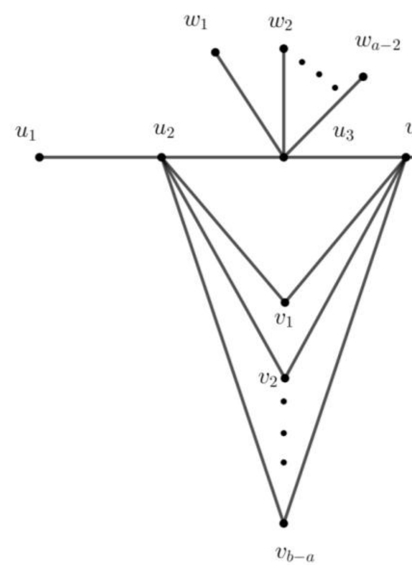

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Fig 2.6

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