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Growth of Maximum Term of Iterated Entire Functions

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ABSTRACT

In this article, we investigate the comparative growth properties of generalize iterated entire functions and maximum term of the related entire functions. Our results improve some earlier results.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS

Let f be any entire function defined in the open complex plane C. Then the maximum modulus $M(r;f)$ of f in the circle $|z|=r$ is defined by $M(r;f)=\max_{|z|=r}|f(z)|$. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ then the maximum term $\mu\,(r;f)$ of f on $|z|=r$ is defined by $\mu\,(r;f)=\max_{n}|a_n|r^n$. We refer to [6,9] and [10] for standard notations and definitions.

Notation 1.1[7] Let $log^{[j]}y=log(log^{[j-1]}y)$, $exp^{[j]}y=exp(exp^{[j-1]}y)$ for any positive integer j, where $log^{[0]}y=y$, $exp^{[0]}y=y$.

Definition 1.2 [9] Let f be an entire function defined in the open complex plane C. We define

$$\rho_f=\lim_{r\to\infty} sup\,\frac{\log\log M(r;f)}{\log r}$$

and

$$\lambda_f=\lim_{r\to\infty} inf\,\frac{\log\log M(r;f)}{\log r}.$$

ρ_f and λ_f are called order and lower order of f respectively.
In[8] we have

$$\mu\,(r;f) \le M(r;f) \le \frac{R}{R-r}\,\mu\,(R;f)$$

For $0 < r < R$. If we take $R = 2r$, then we have for all sufficiently large values of r,

$$\mu\,(r;f) \le M(r;f) \le 2\,\mu\,(2r;f) \qquad (1)$$

Hence

$$\rho_f = \lim_{r\to\infty} sup\, \frac{\log^{[2]}\,\mu(r;f)}{\log r}$$

and

$$\lambda_f = \lim_{r\to\infty} inf\, \frac{\log^{[2]}\,\mu(r;f)}{\log r}.$$

Definition 1.3[1] Let f and g be two entire functions defined in the open complex plane and $0 < c \le 1$. Then for any positive integer $n \ge 2$ we define
$f_g^n(z) = (1-c)g_f^{n-1}(z) + cf(g_f^{n-1}(z))$ and $g_f^n(z) = (1-c)f_g^{n-1}(z) + cg(f_g^{n-1}(z))$,
where $f_g^1(z) = (1-c)z + cf(z)$ and $g_f^1(z) = (1-c)z + cg(z)$.
From the above definition clearly all f_g^n and g_f^n are entire functions.
Many authors like Banerjee and Dutta[2], Dutta[4],[5] prove some results on comparative growth properties of the maximum term of iterated functions.

In this article, we study some comparative growth properties of the maximum term of iterated functions and the maximum term of the related functions. Our results improve and generalize some earlier results.

2. LEMMAS

The following lemmas will be needed in the sequel.

Lemma 2.1[3] Let f and g be any two entire functions defined in the open complex plane C. Then for all sufficiently large values of r,

$$M\left(\frac{1}{8}M\left(\frac{r}{2};g\right) - |g(0)|;f\right) \le M(r;f\circ g) \le M(M(r;g);f).$$

Lemma 2.2 Let f and g be two non-constant entire functions such that $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda_g \le \rho_g < \infty$. Then for any $\vartheta > 0$,

$$\log^{[n]}\mu\left(r;f_g^n\right) \le \begin{cases} (\rho_f + \vartheta)(1+O(1))\log M(r;g) + O(1), & when\ n\ is\ even \\ (\rho_g + \vartheta)(1+O(1))\log M(;,f) + O(1), & when\ n\ is\ odd, \end{cases}$$

for all sufficiently large values of r.

Proof. First we suppose that n is even integer. Then in view of (1) and by Lemma 2.1 it follows that for all sufficiently large values of r,

$$\begin{aligned}\mu\left(r; f_g^n\right) &\leq M\left(r; f_g^n\right)\\ &\leq M\left(r; g_f^{n-1}\right) + M\left(r; f(g_f^{n-1})\right) + O(1)\\ &\leq \left(1 + O(1)\right) M\left(M\left(r; g_f^{n-1}\right); f\right)\\ &\leq 2M\left(M\left(r; g_f^{n-1}\right); f\right)\end{aligned}$$

Hence

$$\log \mu\left(r; f_g^n\right) \leq \log M\left(M\left(r; g_f^{n-1}\right); f\right) + O(1) \leq \left[M\left(r; g_f^{n-1}\right)\right]^{\rho_f + \vartheta} + O(1),$$

$$\begin{aligned}\log^{[2]} \mu\left(r; f_g^n\right) &\leq (\rho_f + \vartheta)\ \log M(r; g(f_g^{n-2})) + O(1)\\ &\leq (\rho_f + \vartheta)\ [M(r; f_g^{n-2})]^{\rho_g + \vartheta} + O(1),\\ \log^{[3]} \mu\left(r; f_g^n\right) &\leq (\rho_g + \vartheta)\ \log M(r; f_g^{n-2}) + O(1),\end{aligned}$$

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$$\begin{aligned}\log^{[n]} \mu\left(r; f_g^n\right) &\leq (\rho_f + \vartheta)\ \log M\left(r; g_f^1\right)) + O(1)\\ &\leq (\rho_f + \vartheta)\ \{\log M(r; z) + \log M(r; g) + O(1)\} + O(1)\\ &\leq (\rho_f + \vartheta)\ (1 + O(1)) \log M(r; g) + O(1).\end{aligned}$$

Similarly if n is odd then for all sufficiently large values of r,

$$\log^{[n]} \mu\left(r; f_g^n\right) \leq (\rho_g + \vartheta)\ (1 + O(1)) \log M(r; f) + O(1).$$

This proves the lemma.

3. THEOREMS

Theorem 3.1 Let f and g be two non-constant entire functions having finite orders and λ_f, $\lambda_g > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$,

(i) $\displaystyle\lim_{r\to\infty} \frac{\{\log^{[n]} \mu(r; f_g^n)\}^{1+\alpha}}{\log \log \mu(\exp(r^p); f)} = 0, if\ p > (1+\alpha)\rho_g\ and\ n\ is\ even,$

(ii) $\displaystyle\lim_{r\to\infty} \frac{\{\log^{[n]} \mu(r; f_g^n)\}^{1+\alpha}}{\log \log \mu(\exp(r^p); f)} = 0, if\ p > (1+\alpha)\rho_f\ and\ n\ is\ odd$

Proof. If $\alpha \leq -1$ then the theorem is trivial. So we suppose that $\alpha > -1$ and n is even. Then from Lemma 2.2 we get for all sufficiently large values of r and any $\vartheta > 0$

$$\log^{[n]} \mu\left(r; f_g^n\right) \leq (\rho_f + \vartheta)(1 + O(1)) \log M(r; g) + O(1) \leq (\rho_f + \vartheta)(1 + O(1)) r^{\rho_g + \vartheta} \qquad (2)$$

Again from Definition 1.2 it follows that for any ϑ $(0 < \vartheta < \lambda_f)$ and for all large values of r,

$$\log\log\mu(\exp(r^p)\,;f) > \left(\lambda_f - \vartheta\right) r^p. \tag{3}$$

So from (2) and (3) we have for all large values of r and any ϑ $(0 < \vartheta < \lambda_f)$,

$$\frac{\{\log^{[n]}\mu\left(r; f_g^n\right)\}^{1+\alpha}}{\log\log\mu(\exp(r^p)\,;f)} \le \left(1 + O(1)\right)\frac{(\rho_f + \vartheta)^{1+\alpha}\, r^{(1+\alpha)(\rho_g+\vartheta)}}{\left(\lambda_f - \vartheta\right) r^p} + o(1).$$

Since $\vartheta > 0$ is arbitrary, we can choose ϑ such that $0 < \vartheta < \min\{\lambda_f, \frac{p}{1+\alpha} - \rho_g\}$.
Hence

$$\lim_{r\to\infty}\frac{\{\log^{[n]}\mu\left(r; f_g^n\right)\}^{1+\alpha}}{\log\log\mu(\exp(r^p)\,;f)} = 0.$$

Similarly we can prove (ii) when n is odd.
This proves the theorem.

Theorem 3.2 Let f and g be two non-constant entire functions having finite orders and λ_f, $\lambda_g > 0$. Then for $p > 0$ and each $\alpha \in (-\infty, \infty)$,

(i) $\lim_{r\to\infty}\frac{\{\log^{[n]}\mu(r; f_g^n)\}^{1+\alpha}}{\log\log\mu(\exp(r^p); g)} = 0, if\ p > (1+\alpha)\rho_f\ and\ n\ is\ odd$

(ii) $\lim_{r\to\infty}\frac{\{\log^{[n]}\mu(r; f_g^n)\}^{1+\alpha}}{\log\log\mu(\exp(r^p); g)} = 0, if\ p > (1+\alpha)\rho_g\ and\ n\ is\ even,$

Proof. If $\alpha \le -1$ then the theorem is trivial. So we suppose that $\alpha > -1$ and n is odd. Then from Lemma 2.2 we get for all sufficiently large values of r and any $\vartheta > 0$

$$\begin{aligned}\log^{[n]}\mu\left(r; f_g^n\right) &\le (\rho_g + \vartheta)(1 + O(1))\log M(r; f) + O(1)\\ &\le (\rho_g + \vartheta)(1 + O(1)) r^{\rho_f + \vartheta} + O(1).\end{aligned} \tag{4}$$

Again from Definition 1.2 it follows that for any ϑ $(0 < \vartheta < \lambda_g)$ and for all large values of r,

$$\log\log\mu(\exp(r^p)\,;g) > \left(\lambda_g - \vartheta\right) r^p. \tag{5}$$

So from (4) and (5) we have for all large values of r and any ϑ $(0 < \vartheta < \lambda_g)$,

$$\frac{\{\log^{[n]}\mu\left(r; f_g^n\right)\}^{1+\alpha}}{\log\log\mu(\exp(r^p)\,;g)} \le \left(1 + O(1)\right)\frac{(\rho_g + \vartheta)^{1+\alpha}\, r^{(1+\alpha)(\rho_f+\vartheta)}}{\left(\lambda_g - \vartheta\right) r^p} + o(1).$$

Since $\vartheta > 0$ is arbitrary, we can choose ϑ such that $0 < \vartheta < \min\{\lambda_g, \frac{p}{1+\alpha} - \rho_f\}$.
Hence

$$\lim_{r\to\infty}\frac{\{\log^{[n]}\mu\left(r; f_g^n\right)\}^{1+\alpha}}{\log\log\mu(\exp(r^p)\,;g)} = 0.$$

Similarly we can prove (ii) when n is even.
This proves the theorem.

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