Journal of Computer and Mathematical Sciences, *Vol.8(3), 76-84 March 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

Bayesian Estimation of Probabilistic Inventory Model with Price Breaks

N. S. Indhumathy[1], P. R. Jayashree[2] and P. Venkatesan[3]

[1]Research Scholar,
Department of Statistics, Presidency College, Chennai-05, INDIA.
[2]Assistant Professor,
Department of Statistics, Presidency College, Chennai-05, INDIA.
[3]Advisor (Research),
KK Child Trust Medical Research Foundation, Chennai, INDIA.

(Received on: March 9, Accepted: March 20, 2017)

ABSTRACT

In this paper, the Baye's estimation of a probabilistic inventory model with exponential lead time distribution along with price breaks is considered. Inventory decisions such as when to order and how much to order for different items of consumption or sales are indeed very important. So, in this model the inventory policy is to order the quantity y, whenever the inventory drops to reorder level r and it is assumed to be a function of the lead time. The aim of this paper is to find the optimal values of ordering quantity and reorder level with price breaks by minimizing the expected total cost. However, the model contains the exponential parameter which is unknown and is estimated through MLE and Baye's under a squared error loss function. The conjugate Gamma prior is used as the prior distribution of exponential distribution. Finally, a numerical MCMC simulation is used to compare the estimators obtained with Expected risk and are shown graphically.

Keywords: Baye's estimator, Expected risk, Exponential distribution, Lead time distribution, Maximum likelihood estimator, Optimum order quantity, Probabilistic demand, Price breaks and Reorder level.

1. INTRODUCTION

Inventory models are classified as either deterministic or stochastic. Deterministic models are models where the demand for a time period is known, whereas in stochastic models the demand is a random variable having a known probability distribution. These models can also be classified by the way the inventory is reviewed, either continuously or periodic. In a continuous model, an order is placed as soon as the stock level falls below the prescribed reorder point. There are several probabilistic inventory models that are available in the literature assuming the demand rate to be the probability distribution rather than the exact value of the demand rate. Haim Shore (1999) provides an optimal solution for stochastic inventory models when the lead-time demand distribution is partially specified. Lu, Song and Zhu (2008) have reviewed the analysis of perishable- inventory systems with censored demand data. Price discounts for increased profitably under partial backordering has given by Matthew J, Drake, Dravid .W Pentico (2011). Joaquin Sicilia, *et al.*,(2012) have discussed an inventory model where backordered demand ratio is exponentially decreasing with the waiting time. In this study we assume that the demand rate is probabilistic in nature with lead time distribution as exponential. The price breaks for the quantity to be ordered is also considered in this model. The objective of the paper is to find the optimal order quantity and reorder level with price breaks by minimizing the expected total cost for a given price break. However, the model contains the exponential parameter which is unknown and is estimated through MLE and Baye's under a squared error loss function.

Maximum likelihood estimation has been the widely used method to estimate the parameter of an exponential distribution. Lately, Bayes method has begun to get the attention of researchers in the estimation procedure. The only statistical theory that combines modelling inherent uncertainty and statistical uncertainty is Bayesian statistics. K.R. Kamath, T.P. M. Pakhala (2002) focus on a Bayesian approach to a dynamic inventory model under an unknown demand distribution. Meghnatisi Z. and N. Nematollahi (2009) have discussed inadmissibility of usual and mixed estimators of two ordered gamma scale parameters under reflected gamma loss function. Kaminska A (2010) has given the equivalence of bayes and robust bayes estimators for various loss functions. So, in this paper the conjugate Gamma prior is used as the prior distribution for parameter of exponential distribution and the parameter is also estimated through MLE. A numerical MCMC simulation is used to compare the estimators obtained by MLE and Baye's under expected risk and are shown graphically.

2. NOTATIONS OF THE MODEL

c_1 - Purchasing cost per unit time if $y \leq q$
c_2 - Purchasing cost per unit time if $y > q$
d - Expected demand per unit time
h - Holding cost per Inventory unit per unit time
p - Shortage cost per Inventory unit
k - Setup cost per order
f(x) - Exponential Lead time distribution - $\lambda e^{-\lambda x}$
$1/\lambda$ –Average Time between placing and receiving orders
r - Reorder level
q - Price breaks quantity
y- Order quantity
The Total cost function are determined as the

1. Average Inventory cost per unit time is hI where I= y/2+ {r-E(x)}.
2. Shortage occurs when x>r and the expected shortage quantity per cycle is $s=\int_r^{\infty}(x-r)$ f(x). The Expected Shortage cost per unit time is pds/y.

It may be offered at q discount that depends on the size of the order and the Purchasing cost per unit time $=\begin{cases} dc_1, y \le q \\ dc_2, y > q \end{cases}$ where $c_1 > c_2$

3. The approximate number of orders per unit time is d/y and therefore the Setup cost per unit time is kd/y

3. DESCRIPTION OF THE MODEL

In this model probabilistic nature of demand with the exponential lead time distribution is considered. The shortage cost per inventory unit is also considered. In purchasing cost, the price breaks for the quantity to be ordered is also incorporated into the model. The Inventory policy calls for ordering the quantity y whenever the inventory drops to level r. The Reorder level r is a function of the lead time between placing and receiving an order. The total cost function with price breaks involving setup, holding, purchasing, and shortage costs are given by

$$TCU_{1(y,r)} = \frac{kd}{y} + h\left[\frac{y}{2} + r - \frac{1}{\lambda}\right] + \frac{pd}{y}\int_r^{\infty}(x-r)\,\lambda\, e^{-\lambda x}dx + dc_1 \quad y \le q \tag{1}$$

$$TCU_{2(y,r)} = \frac{kd}{y} + h\left[\frac{y}{2} + r - \frac{1}{\lambda}\right] + \frac{pd}{y}\int_r^{\infty}(x-r)\,\lambda\, e^{-\lambda x}dx + dc_2 \quad y > q \tag{2}$$

Solving the above equations (1) and (2) we get

$$TCU_{1(y,r)} = \frac{kd}{y} + h\left[\frac{y}{2} + r - \frac{1}{\lambda}\right] + \frac{pd}{y}\frac{e^{-r\lambda}}{\lambda} + dc_1 \quad y \le q \tag{3}$$

$$TCU_{2(y,r)} = \frac{kd}{y} + h\left[\frac{y}{2} + r - \frac{1}{\lambda}\right] + \frac{pd}{y}\frac{e^{-r\lambda}}{\lambda} + dc_2 \quad y > q \tag{4}$$

By differentiating the above equations (3) , (4) and equating to zero we get

$$\frac{\partial TCU_{1(y,r)}}{\partial y} = \frac{-kd}{y^2} + \frac{h}{2} - \frac{pde^{-r\lambda}}{\lambda y^2} = 0 \tag{5}$$

$$\frac{\partial TCU_{2(y,r)}}{\partial r} = h - \left(\frac{pd}{y}\right)\left(e^{-r\lambda}\right) = 0 \tag{6}$$

Solving equations of (5) and (6) to get the optimal values of y and r we get

$$y^* = \sqrt{kd - h/2 + pd\,\frac{e^{-r\lambda}}{\lambda}}\,. \tag{7}$$

$$r^* = \frac{1}{\lambda}\log\frac{pd}{hy^*}\,. \tag{8}$$

y^* and r^* cannot be determined directly from the above equations (7) and (8) . So, the following steps are used to find the solutions of y and r.

(a) Use the initial solution $r_0=0$ in equation (7) to get $y_1 = y^* = \sqrt{kd - \frac{h}{2} + \frac{pd}{\lambda}}$ and go to (b)

(b)_ Use y^* to determine r from equation (8). If $r_i=r_{i-1}$,i=0,1,2,... then stop and the optimal solution is $y^*=y_i$, and $r^*=r_i$, otherwise use r_i in Equation(7) to compute y^* then set i=i+1 and repeat (a)

The above two steps are repeated until it converges in a finite number of iterations, provided a feasible solution exists. Thus the values of y^* and r^* are obtained by using the above steps. The determination of the optimal order quantity also depends on the price break point q. Therefore the optimal order quantity with price breaks is given by

$$y = \begin{cases} y^*, & \text{if } q < y^*, \\ q, & \text{if } q > y^* \end{cases}$$

4. PARAMETER ESTIMATION

MAXIMUM LIKLIHOOD ESTIMATION

The probability density function of the exponential distribution is given by,

$$f(x\,;\lambda) = \begin{cases} \lambda e^{-\lambda x} & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases} \tag{1}$$

Suppose $X_1,X_2,\ldots,X_n$ is a random sample from exponential distribution (1). Let $(x_1,x_2,\ldots,x_n)$ be the observed values of $(X_1,X_2,\ldots,X_n)$, then the likelihood function based on $(x_1,x_2,\ldots,x_n)$ is given by,

$$L(\lambda/X) = \prod_{i=1}^{n} f(x_i,\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x} = \lambda^n e^{-\lambda\sum_{i=1}^{n} x_i}$$

To calculate the maximum likelihood estimator, the natural logarithm of likelihood function is maximised i.e. differentiating with respect to λ and equating each result to zero.

$$\frac{d}{d\lambda}\left(\ln L(\lambda/X)\right) = n\,\ln\,\lambda - \sum_{i=1}^{n} x_i = 0$$

The MLE of λ given by $\frac{n}{\sum_{i=1}^{n} x_i}$

BAYES ESTIMATION

In this section, we consider the Bayes estimation for the parameter λ assuming the conjugate of prior distribution for λ as two parameter Gamma distribution given as

$$f(\lambda/\alpha,\beta)=\begin{cases}\frac{\beta^{\alpha}}{\gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta x} & ,\lambda\geq 0\\ 0 & ,\lambda<0\end{cases} \qquad \alpha>0,\ \beta>0$$

The likelihood function is assumed as L(λ/x) and the posterior distribution is,

$$p(\lambda/x)\propto L(\lambda/x)\ \ f(\alpha,\beta)$$

$$p(\lambda/X)\propto(\lambda^{n}\ e^{-\lambda\sum_{i=1}^{n}x_i})(\lambda^{\alpha-1}e^{-\beta})$$

$$p(\lambda/X)\propto\lambda^{n+\alpha-1}\ e^{-\lambda[\beta+\sum_{i=1}^{n}x_i]}$$

This follows Gamma distribution with parameter $\gamma(n+\tilde{\alpha}\ \ ,\tilde{\beta}+\sum_{i=1}^{n}x_i)$

The mean and variance are given by

Mean= $\frac{\alpha}{\beta}=\frac{n+\tilde{\alpha}}{\tilde{\beta}+\sum_{i=1}^{n}x_i}$

Variance= $\frac{\tilde{\alpha}}{\tilde{\beta}^2}$

The parameter is estimated from the above methods and using this in order quantity and reorder level functions we get

Optimal order quantity $y^* = \sqrt{kd-h/2+pd\,\frac{e^{-r\lambda}}{\hat{\lambda}}}$

Optimal reorder level $r^*=\frac{1}{\hat{\lambda}}\log\frac{pd}{hy^*}$

NUMERICAL SIMULATION

To compare the different estimators of the parameter λ of the exponential distribution, the Expected risks under squared error loss of the estimates are considered. These estimators are obtained by maximum likelihood and Bayes methods . The MCMC procedure for estimation is as follows

(i) A sample of size n is then generated from the density of the exponential distribution, which is considered to be the informative sample.

(ii) The MLE and Bayes estimators are calculated with $\alpha = n + \tilde{\alpha}$, $\beta = \tilde{\beta} + \sum_{i=1}^{n} x_i$

(iii) Steps (i) to (ii) are repeated N = 2000 times for different sample sizes and the risks under squared error loss of the estimates are computed by using:

Expected Risk ($\hat{\lambda}$)= $\frac{1}{N}\sum_{i=1}^{n}\left(\tilde{\lambda}_i - \lambda\right)$ Where, $\hat{\lambda}_i$ is the estimate at the i^{th} run

Assuming the value of λ = 0.1 , the estimated value of $\hat{\lambda}$ using MLE and Baye's along with Expected risk are given in Table 1.

It is seen that for small sample sizes the estimators under the Expected Loss function have smaller ER when choosing proper parameters α and β. But for larger sample sizes (n>50), all the estimators have approximately same ER. The obtained results are given in Table 1 and shown graphically in Figure 1 and 2. Therefore the estimated value of λ is $\hat{\lambda} = 0.09$.

Table:1 Parameter Estimation and Expected Risk

n	Criteria	$\alpha = n + \tilde{\alpha}$ and $\beta = \tilde{\beta} + \sum_{i=1}^{n} x_i$			
		MLE	**α=0.5, β=0**	**α=1, β=0.5**	**α=1.5, β=1**
10	Estimated value ER	0.1408 0.00041	0.1296 0.00022	0.1379 0.00031	0.1402 0.0003
25	Estimated value ER	0.1142 0.00008	0.1045 0.00007	0.1015 0.00006	0.1024 0.00007
50	Estimated value ER	0.1106 0.00007	0.1006 0.00006	0.1015 0.00007	0.1024 0.00005
75	Estimated value ER	0.1098 0.00005	0.0996 0.00005	0.1002 0.00004	0.1008 0.00004
100	Estimated value ER	0.1039 0.000004	0.0945 0.000008	0.0950 0.000008	0.0954 0.000008
125	Estimated value ER	0.1050 0.000004	0.0954 0.000005	0.0957 0.000005	0.0960 0.000005
150	Estimated value ER	0.0960 0.000005	0.0937 0.000005	0.0940 0.000005	0.0943 0.000004
200	Estimated value ER	0.0950 0.000004	0.0943 0.000004	0.0940 0.000004	0.0937 0.000004

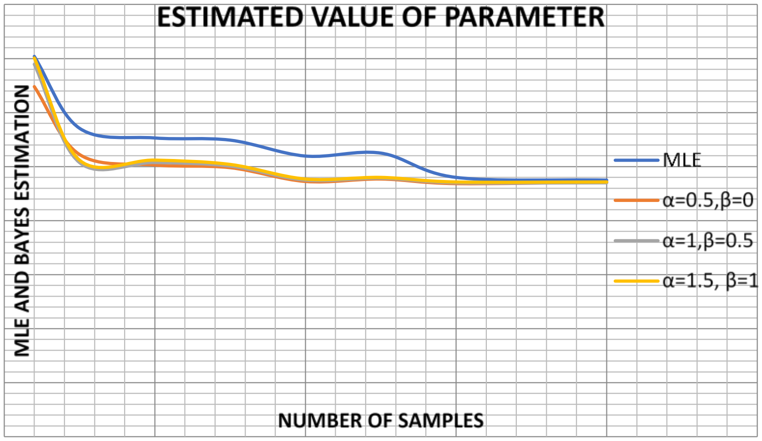

FIGURE:1 MLE AND BAYES ESTIMATION

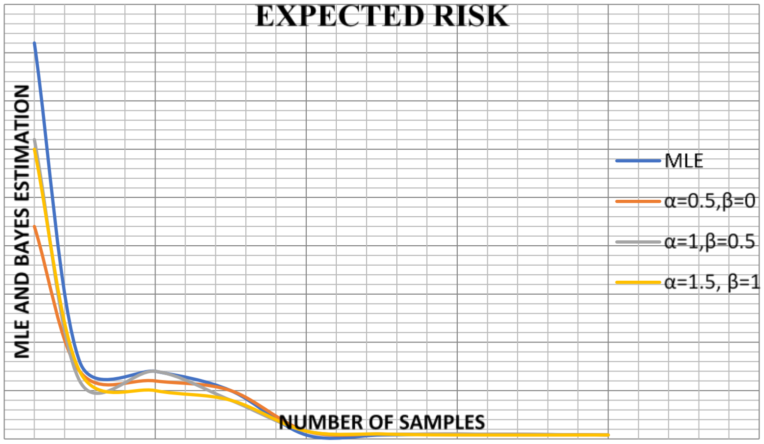

FIGURE:2 EXPECTED RISK UNDER LOSS FUNCTION

5. NUMERICAL ILLUSTRATION

Using the estimated value of Baye's , $\widehat{\lambda} = 0.094$ the optimal order quantity with different price breaks q= 50, 75, 100, 150, 200, 300 are shown in Table 2. By taking the values of other constants as d=1000, h=3, p=50, k=10 , c_1=3, c_2=2.50 the following table gives the optimal order quantity with price breaks and is shown graphically in Fig.3.

$\widehat{\lambda}$	y^*	r^*	Price break q	Optimal order quantity with price break y^*
0.09	124.7911	51.1871	50	124.7911
0.09	124.7911	51.1871	75	124.7911
0.09	124.7911	51.1871	100	124.7911
0.09	124.7911	51.1871	150	150
0.09	124.7911	51.1871	200	200
0.09	124.7911	51.1871	300	300

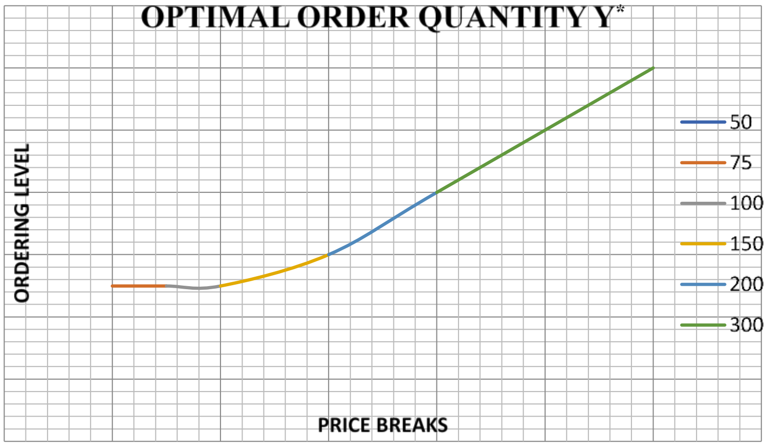

FIGURE:3 OPTIMAL ORDER QUANTITY WITH PRICE BREAKS

6. CONCLUSION

The probabilistic inventory model with exponential lead time distribution along with price breaks is considered in this paper. The MLE and Baye's estimation are used to estimate the parameter of exponential distribution. Based on the estimated value of Baye's, the optimal order quantity and reorder level with price breaks are found for the model. The model is also illustrated numerically and shown graphically.

REFERENCES

1. Chakrabarthy, T. and Chauduri, K.S. An EOQ Model for items with Weibull distribution, shortages are trended demand. An extension of Philip's model. *Computers and operational research*. 25,649-657 (1998).
2. Haim Shore. Optimal solutions for stochastic inventory models when the lead–time demand distribution is partially specified. *International journal of production economics*. 59(1-3): 477-485 (1999).
3. Joaquin Sicilia, Luis A. San-Jose, Juan Garcia-Lagura. An inventory model where back ordered demand ratio is exponentially decreasing with the waiting time. *Annals of operations research*. Vol-199, 1:137-155 (2012).
4. Kamath, K.R. Pakhala, T.P.M A Bayesian approach to a dynamic inventory model under an unknown demand distribution. *Computers and operations research*. 29:403-422 (2002).
5. Kamińska, A. The equivalence of Bayes and robust bayes estimators for various loss functions. *Stat. Papers*. 51: 179-191 (2010).
6. Lu, Song, and Zhu. Analysis of perishable-Inventory systems with censored demand data. *Operations Research.* 56(4), 1034-1038 (2008).
7. Matthew J. Drake, Dravid W. Pentico. Price discounts for increased profitably under partial backordering. *International Transaction in operations research*. 18:1,87-101 (2011).

8. Meghnatisi,Z. and Nematollahi, N. Inadmissibility of usual and mixed estimators of two ordered gamma scale parameters under reflected gamma loss function. *J. Math. Extension*. 3(2): 89-99 (2009).
9. Moncer Hariga, Mohamed Ben-Daya. Some stochastic Inventory model with deterministic variable lead time, *European Journal of Operational Reasearch*,113(1), 42-51 (1999).
10. Nita H. Shah. Probabilistic time- scheduling model for exponentially decaying inventory when delays in payments are permissible, *International Journal of Production Economics*, 32,77-82 (1993).
11. Samak Kulkarni, S.M. and Rajhans, N.R. Determination of Optimum Inventory model for minimizing total inventory cost. *ELSEVIER*. 51,803-809 (2013).
12. Stephen, C. H., Leung, Zhong Yoa. Stochastic Inventory model with uniformly distributed demand for ordering or returning policies in single periods. *Computers and Engineering*. 61, 242-253 (2011).