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Study of Thermoelastic Deformation of a Solid Circular Cylinder by Application of Fractional Order Theory

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ABSTRACT

The present work deals with analysis of thermoelastic deformation of a solid circular cylinder by application of fractional order theory. Here an attempt is made to determine the unknown temperature, displacement and stress function and deflection on outer curved surface of a solid circular cylinder occupying the space $D: 0 \leq r \leq a, 0 \leq z \leq h$ with the lower and upper surface kept thermally insulated. The finite Fourier sine transform and Laplace transform techniques have been used to solved the problem under consideration.

Keywords: Fractional Order Theory, solid circular cylinder, Fourier sine transform, Laplace transform, Deflection, stresses.

INTRODUCTION

Variation of time-fractional differential operators with memory effects was investigated by Povstenko[1,2]. Time-fractional heat conduction in a composite medium is solved analytically for an infinite matrix and is presented for a spherical inclusion by Povstenko in[3]. Associated Thermal Stresses is determined in space with a Source which Varying Harmonically in Time Space in context of Fractional Heat Conduction.

Lamba[11] determined the temperature distribution, unknown temperature gradient, displacement, stress functions and thermal stresses on the outer curved surface of a thin annular fin with known interior heat flux. Khobragade[6] discussed an inverse axially symmetric quasi-static problem of thermoelasticity for a thin clamped circular plate in which a heat flux is prescribed on an internal cylindrical surface of the plate with the help of a generalized integral transform technique. Lamba[4] studied the three-dimensional inverse transient thermoelastic problem for a thin rectangular object within the context of the theory of generalized thermoelasticity. Tikhe[12] determined the unknown heating temperature and temperature distributions on the upper surface of a thin circular plate. Lamba[5] studied the uncoupled thermoelastic response of thick cylinder of length 2h in which heat sources are generated according to the linear function of the temperature, with boundary conditions of the radiation type. Raslan[13] applied the fractional order theory of thermoelasticity to the two dimensional problem of a thick plate whose lower and upper surfaces are traction free and subjected to a given axisymmetric temperature distribution.

Here an attempt is made to determine the unknown temperature, displacement and stress function and deflection on outer curved surface of a solid circular cylinder occupying the space $D: 0 \le r \le a, 0 \le z \le h$ with the lower and upper surface kept thermally insulated. The finite Fourier sine transforms and Laplace transform techniques have been used to find the solution of the problem.

NOTATION AND GOVERNING EQUATIONS

(a) The basic relationship for the problem defined above can be summarized as follows: The expression for displacement function is defined by[13] as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{1+v}{1-v}\right) a_t T \tag{1}$$

Where $\phi = 0$ at $r = a$ and $r = b$ for all time t (2)

Initially $U = \theta = \sigma_{rr} = \sigma_{\theta\theta} = 0 \quad at \quad t = 0$ (3)

Here v is Poisson's ratio, a_t is coefficient of thermal expansion and T denotes temperature for the circular plate.

(b) The Caputo type fractional derivative for nonlocal heat conduction is defined by[12]

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \qquad n-1 < \alpha < n \tag{4}$$

To find Laplace transforms of the Caputo derivative it needs to know the initial values of the function $f(t)$ and its integer derivatives of the order $r = 0,1,2,....,n-1$

$$L\left\{\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{r=0}^{r=n-1} f^{(r)}(0^{+}) s^{\alpha-1-r}, \qquad n-1 < \alpha < n \tag{5}$$

(c) The radial and axial displacement U and W satisfy the uncoupled thermoelastic equation as (Sierakowski and sun 1968) are

$$\nabla^2 U - \frac{U}{r^2} + (1-2v)^{-1}\frac{\partial e}{\partial r} = 2\left(\frac{1+v}{1-2v}\right)a_t\,\frac{\partial T}{\partial r} \tag{6}$$

$$\nabla^2 W + (1+2v)^{-1}\,\frac{\partial e}{\partial r} = 2\left(\frac{1+v}{1-2v}\right)a_t\,\frac{\partial T}{\partial z} \tag{7}$$

where $e = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z}$ is the volume dilation and

$$U = \frac{\partial \phi}{\partial r} \tag{8}$$

$$W = \frac{\partial \phi}{\partial z} \tag{9}$$

(d) The thermal stress components in terms of the displacement components are given as,

$$\sigma_r = (\lambda + 2G)\frac{\partial U}{\partial r} + \lambda\left(\frac{U}{r} + \frac{\partial W}{\partial z}\right) \tag{10}$$

$$\sigma_z = (\lambda + 2G)\frac{\partial W}{\partial z} + \lambda\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) \tag{11}$$

$$\sigma_\theta = (\lambda + 2G)\frac{U}{r} + \lambda\left(\frac{\partial W}{\partial z} + \frac{\partial U}{\partial r}\right) \tag{12}$$

$$\tau_{rz} = G\left(\frac{\partial W}{\partial r} + \frac{\partial U}{\partial z}\right) \tag{13}$$

where $\lambda = \frac{2Gv}{1-2v}$ is the Lame's constants, G is the shear modulus and U and W are the displacement components.

(e) The deflection $W(r,t)$ of a clamped plate satisfy the differential equation is

$$D\nabla_1^4 W(r,t) = -\frac{\nabla_1^2 M_1(r,t)}{(1-v)} \tag{14}$$

where $v =$ Poisson's ratio of the material of the cylinder,

$$\nabla_1^2 = \frac{d^2}{dr^2} + \frac{1}{r}\,\frac{d}{dr}$$

$$M_T(r,t) = \alpha E \int_0^h z.T(r,z,t)dz \tag{15}$$

with $W(r,t) = 0$ at $r = a$ (16)

FORMULATION OF THE PROBLEM

Consider a solid cylinder of length h occupying the space $D: 0 \le r \le a$, $0 \le z \le h$ in the Application of Fractional Order Theory. Initially the cylinder is at zero temperature. The lower and upper surface is kept thermally insulated.

The heat conduction equation in time fractional order context for solid circular cylinder is given as[15]

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial^\alpha T}{\partial t^\alpha} \; ; \quad 0 \le r \le a, \quad 0 \le z \le h \tag{17}$$

subject to the initial condition

$$T(r,z,t) = 0 \tag{18}$$

the boundary conditions

$$T(a,z,t) = g(z)\delta(t) \tag{19}$$

$$\left[T(r,z,t)\right]_{z=0} = 0 \tag{20}$$

$$\left[T(r,z,t)\right]_{z=h} = 0 \tag{21}$$

$$T(r,z,t) = f(z)\delta(t) \, , 0 < r < a \tag{22}$$

where k is the thermal diffusivity of the material of the plate.

The equations (17) to (22) constitute the mathematical formulation of the problem under consideration.

SOLUTION OF THE PROBLEM

Applying finite Fourier sine transform defined in to the equations (17) to (19), (22) and using (20), (21) one obtains

$$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} - p^2 \overline{T} = \frac{1}{k}\frac{\partial^\alpha \overline{T}}{\partial t^\alpha} \tag{23}$$

$$\overline{T}(r,m,o) = 0 \tag{24}$$

$$\overline{T}(a,m,t) = \overline{g}(m)\delta(t) \tag{25}$$

$$\overline{T}(r,m,t) = \overline{f}(m)\delta(t) \tag{26}$$

where $p^2 = \left(\frac{m\pi}{h}\right)^2$ (27)

where $\overline{T}$ is the Fourier sine transform of T and m is the Fourier sine transform parameter.
Applying Laplace transform to the equations (23), (25), (26) and using (24) one obtains

$$\frac{\partial^2 \overline{T}^*}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}^*}{\partial r} - q^2 \overline{T}^* = 0 \tag{28}$$

$$q^2 = p^2 + \frac{1}{k}\left[s^\alpha L\{\theta\} - \sum_{r=0}^{r=n-1} \theta^{(r)}(0^+) s^{\alpha-1-r}\right] \tag{29}$$

$$\overline{T}^*(a, m, s) = \overline{g}^*(m) \tag{30}$$

$$\overline{T}^*(r, m, s) = \bar{f}^*(m) \tag{31}$$

Equation (28) is a second order differential equation whose solution is given by

$$\overline{T}^* = A\, I_0(qr) + B\, K_0(qr) \tag{32}$$

As $r \to 0$, $K_0(qr) \to \infty$. But by physical consideration $\overline{T}^*$ remains finite. Therefore *B* must be zero.
Using equation (31) in (32) one obtains

$$A = \frac{\bar{f}^*(m)}{I_0(qr)}, \quad B = 0 \tag{33}$$

Substituting the values of *A* and *B* in (32) one obtains

$$\overline{T}^* = \frac{\bar{f}^*(m)}{I_0(qa)} \cdot I_0(qr) \tag{34}$$

Applying inverse Laplace transform to the equation (34) one obtains

$$\overline{T}(r, m, t) = \sum_{n=1}^{\infty} \left(\frac{2k}{r}\right) \lambda_n \cdot \frac{J_0(\lambda_n r)}{J_1(\lambda_n r)} \bar{f}(m) E_\alpha\left(-k\left(p^2 + \lambda_n^2\right)(t^\alpha - t'^\alpha)\right) \tag{35}$$

Applying inverse finite Fourier sine transform to the equation (11.2.13) one obtains

$$T(r, z, t) = \frac{2}{h}\sum_{m=1}^{\infty} \frac{\sin m\pi z}{h} \sum_{n=1}^{\infty} \left(\frac{2k}{r}\right) \lambda_n \frac{J_0(\lambda_n r)}{J_1(\lambda_n r)} \bar{f}(m) E_\alpha\left(-k\left(p^2 + \lambda_n^2\right)(t^\alpha - t'^\alpha)\right) \tag{36}$$

Here $E_\alpha(.)$ represents Mittag-Leffler function.

DETERMINATION OF THERMOELASTIC DISPLACEMENT

Substituting the value of (36) in (1) one obtains

$$\phi(r,z,t)=-\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\frac{\lambda_n}{(p^2+\lambda_n^2)}\,\frac{J_0(\lambda_n r)}{J_1(\lambda_n r)}\,\bar{f}(m)E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right) \tag{37}$$

$$U=\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\frac{\lambda_n{}^2}{(p^2+\lambda_n^2)}\,\frac{J_1(\lambda_n r)}{J_1(\lambda_n a)}\,\bar{f}(m)E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right) \tag{38}$$

$$W=-\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\cos\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\frac{\lambda_n p}{(p^2+\lambda_n^2)}\,\frac{J_0(\lambda_n r)}{J_1(\lambda_n r)}E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right) \tag{39}$$

DETERMINATION OF STRESS FUNCTIONS

Substituting the value of (38), (39) in (10) to (13) one obtains

$$\begin{aligned}\sigma_r&=(\lambda+2G)\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\frac{\lambda_n{}^3}{(p^2+\lambda_n^2)}\,\frac{J_1'(\lambda_n r)}{J_1(\lambda_n a)}E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right)\\&\quad+\lambda\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\left[\frac{\lambda_n{}^2}{r(p^2+\lambda_n^2)}\frac{J_1(\lambda_n r)}{J_1(\lambda_n a)}+\frac{\lambda_n p^2}{(p^2+\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}\right]\\&\quad\times E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right)\end{aligned} \tag{40}$$

$$\begin{aligned}\sigma_z&=(\lambda+2G)\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\frac{\lambda_n p^2}{(p^2+\lambda_n^2)}\,\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right)\\&+\lambda\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\left[\frac{\lambda_n{}^3}{(p^2+\lambda_n^2)}\frac{J_1'(\lambda_n r)}{J_1(\lambda_n a)}+\frac{\lambda_n{}^2}{r(p^2+\lambda_n^2)}\frac{J_1(\lambda_n r)}{J_1(\lambda_n a)}\right]\\&\quad\times E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right)\end{aligned} \tag{41}$$

$$\begin{aligned}\sigma_\theta&=(\lambda+2G)\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\frac{\lambda_n{}^2}{r(p^2+\lambda_n^2)}\,\frac{J_1(\lambda_n r)}{J_1(\lambda_n a)}E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right)\\&+\lambda\left(\frac{1+v}{1-v}\right)a_t\cdot\frac{2}{h}\sum_{m=1}^{\infty}\sin\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\left[\frac{\lambda_n p^2}{(p^2+\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}+\frac{\lambda_n{}^3}{(p^2+\lambda_n^2)}\frac{J_1'(\lambda_n r)}{J_1(\lambda_n a)}\right]\\&\quad\times E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right)\end{aligned} \tag{42}$$

$$\tau_{rz}=\frac{4G}{h}\left(\frac{1+v}{1-v}\right)a_t\sum_{m=1}^{\infty}\cos\frac{m\pi z}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{r}\right)\frac{\lambda_n{}^2 p}{(p^2+\lambda_n^2)}\,\frac{J_1(\lambda_n r)}{J_1(\lambda_n a)}E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha-t'^\alpha)\right) \tag{43}$$

DETERMINATION OF $M_T(r, t)$

Substituting the value of $T(r, z, t)$ in the equation (15) one obtains

$$M_T(r,t) = 2\alpha E \sum_{m=1}^{\infty} (-1)^{m+1} \frac{h}{m\pi} \sum_{n=1}^{\infty} \left(\frac{2k}{r}\right) \lambda_n \frac{J_0(\lambda_n r)}{J_1(\lambda_n r)} E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha - t'^\alpha)\right) \tag{44}$$

DETERMINATION OF THERMOELASTIC DEFLECTION

Using the equation (44) in the equation (14) one obtains

$$W(r,t) = \frac{4\alpha E k}{D(\nu-1)r} \sum_{m=1}^{\infty} (-1)^m \frac{h}{m\pi} \sum_{n=1}^{\infty} \left[\frac{J_0(\lambda_n r) - J_0(\lambda_n a)}{\lambda_n J_1(\lambda_n r)}\right] E_\alpha\left(-k\left(p^2+\lambda_n^2\right)(t^\alpha - t'^\alpha)\right) \tag{45}$$

CONCLUSIONS

In this work, the temperature, displacement, thermal stresses and deflection has been determined for a clamped circular plate on a solid circular cylinder in context of fractional order theory of thermoelasticity. The finite Fourier sine transforms and Laplace transform techniques have been used to solve the problem. The system of equations proposed in this study can be adapted to design of useful structures or machines in engineering applications.

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