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Thermal Deflection and Stresses of a Circular Disk Due to Partially Distributed Heat Supply by Application of Fractional Order Theory

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ABSTRACT

The present work deals with the Mathematical Modelling of Thermoelastic thin Hollow Circular Disk occupying the space $D:\ a \le r \le b;\ -h/2 \le z \le h/2$ by the Application of Fractional Order Theory. The initial temperature of the disk and the temperature of the surrounding medium are assumed to be same, which is kept constant. The upper surface of disk is subjected to axisymmetric and partially distributed heat supply, while the lower surface is kept thermally insulated. The attempt made to analyze the thermal stresses and deflection of Disk mathematically by using Integral transform technique.

Keywords: Fractional Order Theory, hollow circular disk, Fourier transform, Hankel transform, Laplace transform, Deflection, stresses.

INTRODUCTION

Now a day's fractional calculus has become popular new mathematical method of solution of diverse problems in mathematics, science, and engineering. It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations. Fractional calculus has many applications in differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and electrochemistry.

Variation of time-fractional differential operators with memory effects was investigated by Povstenko[1,2]. Time-fractional heat conduction in a composite medium is solved analytically for an infinite matrix and is presented for a spherical inclusion by Povstenko in[3]. Associated Thermal Stresses is determined in space with a Source which Varying Harmonically in Time Space in context of Fractional Heat Conduction.

Lamba[4] determined the thermal deflection of a thin circular plate subjected to radiation type boundary condition by using integral transform technique. Kumar[5] investigated the thermoelastic problems in a nonhomogeneous thick annular disc due to partial heating and boundary conditions of the radiation type. Lamba[6] studied the three-dimensional inverse transient thermoelastic problem for a thin rectangular object within the context of the theory of generalized thermoelasticity. Lamba[7] studied the uncoupled thermoelastic response of thick cylinder of length *2h* in which heat sources are generated according to the linear function of the temperature, with boundary conditions of the radiation type. Khobragade[8] discussed an inverse axially symmetric quasi-static problem of thermoelasticity for a thin clamped circular plate in which a heat flux is prescribed on an internal cylindrical surface of the plate with the help of a generalized integral transform technique. Kedar[9] analyzed the heat conduction and thermal stresses in a hollow cylinder with non homogeneous material properties. Kamdi[10] determined displacement function and thermal stresses of a finite length isotropic functionally graded hollow cylinder subjected to uniform temperature field. Kedar[11] determined the heat conduction and thermal stresses of a hollow cylinder with inhomogeneous material properties and internal heat generation. Warbhe *et al.*,[12] studied quasi-static approach for fractional-order theory of thermoelasticity in two-dimensional problem of a thin circular plate.

Notation and governing equations

(a) The basic relationship for the problem defined above can be summarized as follows: The expression for displacement function is defined by[13] as

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+\nu)\, a_t\, \theta \tag{1}$$

Where $\frac{\partial U}{\partial r}$ at $r = a$ and $r = b$ for all time t (2)

Initially $U = \theta = \sigma_{rr} = \sigma_{\theta\theta} = 0 \quad at \quad t = 0$ (3)

Here ν is Poisson's ratio, a_t is coefficient of thermal expansion and θ denotes temperature for the circular disk.

The stress components σ_{rr} and $\sigma_{\theta\theta}$ of the circular disk are given by

$$\sigma_{rr} = -\frac{2\mu}{r}\frac{\partial U}{\partial r}, \quad \sigma_{\theta\theta} = -2\mu\frac{\partial^2 U}{\partial r^2} \tag{4}$$

In the plane state of stress within the circular disk

$$\sigma_{rz} = \sigma_{zz} = \sigma_{\theta z} = 0 \tag{5}$$

(b) The Caputo type fractional derivative for nonlocal heat conduction is defined by[12]

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau , \qquad n-1 < \alpha < n \quad (6)$$

To find Laplace transforms of the Caputo derivative it needs to know the initial values of the function $f(t)$ and its integer derivatives of the order $P = 0,1,2,....,n-1$

$$L\left\{\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{P=0}^{P=n-1} f^{(P)}(0^{+})s^{\alpha-1-P}, \qquad n-1<\alpha<n \quad (7)$$

(c) The differential equation satisfying the deflection function $w(r,t)$ as defined in[14] is given by

$$\nabla^4 \omega = \frac{\nabla^2 M_T}{D(1-v)} \quad (8)$$

where M_T is the thermal moment of the disk defined as

$$M_{\theta} = a_t E \int_{-h/2}^{h/2} \theta(r,z,t) z \quad dz , \quad (9)$$

D is the flexural rigidity of the disk denoted as

$$D = \frac{Eh^3}{12(1-v^2)} \quad (10)$$

a_t, E and v are the coefficients of the linear thermal expansion, Young's modules and Poisson's ratio of the disc material respectively and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad (11)$$

Since the edge of disk is fixed and clamped,

$$\omega = \frac{\partial \omega}{\partial r} = 0 \text{, at } r = 0,\ a . \quad (12)$$

Initially $T = \omega = 0$, at $t = 0$ (13)

(d) The thermal stress components in terms of the resultant moments and shearing forces are given in[14] as,

$$M_{rr} = -D\left(\frac{\partial^2 w}{\partial r^2} + v\frac{1}{r}\frac{\partial w}{\partial r}\right) - \frac{1}{(1-v)} M_{\theta} \quad (14)$$

$$M_{TT} = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \frac{1}{(1-v)} M_{\theta} \quad (15)$$

$$Q_r = -D\frac{\partial}{\partial r}\left(\nabla^2 w\right) - \frac{1}{(1-\nu)}\frac{\partial M_\theta}{\partial r} \tag{16}$$

$$M_{r\theta} = Q_\theta = 0 \tag{17}$$

FORMULATION OF THE PROBLEM

We consider a Thermoelastic problem of a thin Hollow Circular Disk occupying the Space $D:\ a \le r \le b;\ -h/2 \le z \le h/2$ in the Application of Fractional Order Theory. The initial temperature of the disk and the temperature of the surrounding medium is assumed to be same, which is kept constant. The upper surface $z = h/2$ of disk is subjected to axisymmetric and partially distributed heat supply $-\frac{Q_0}{\lambda} f(r)\,\delta(t)$, while the lower surface $z = -h/2$ is kept thermally insulated.

For heating process the heat conduction equation in time fractional order context for circular disk is given as[15]

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{k}\frac{\partial^\alpha\theta}{\partial t^\alpha}\ ;\quad a \le r \le b,\quad -h/2 \le z \le h/2 \tag{18}$$

with initial condition and boundary conditions

$$\theta = 0 \quad t = 0,\ 0 < \alpha < 2 \tag{19}$$

$$\left.\frac{\partial\theta}{\partial r}\right|_{r=a} = 0\ ;\ \ t > 0 \tag{20}$$

$$\left.\frac{\partial\theta}{\partial r}\right|_{r=b} = 0\ ;\ \ t > 0 \tag{21}$$

$$\left.\frac{\partial\theta}{\partial z}\right|_{z=h/2} = -\frac{Q_0}{\lambda} f(r)\,\delta(t)\ ;\ \ t > 0 \tag{22}$$

$$\left.\frac{\partial\theta}{\partial z}\right|_{z=-h/2} = 0\ ;\ \ t > 0 \tag{23}$$

Here θ is Temperature change, k is thermal diffusivity and λ denotes thermal conductivity for thin hollow circular disk.

Finite Hankel transform H over the variable r and its inverse transform for the temperature function $\theta(r,z,t)$, defined in[16] as

$$\bar{\theta}(\beta_m, z, t) = \int_{r'=a}^{b} r'\, K_0(\beta_m, r')\, \theta(r', z, t)\, dr' \tag{24}$$

$$T(r,z,t)=\sum_{m=1}^{\infty}K_0(\beta_m,r)\bar{\theta}(\beta_m,z,t) \tag{25}$$

Where

$$K_0(\beta_m,r)=\frac{\pi}{\sqrt{2}}\frac{\beta_m\ J_0'(\beta_m b)Y_0'(\beta_m b)}{\left[1-\frac{J_0'(\beta_m b)}{J_0'(\beta_m a)}\right]}\left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)}-\frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}\right] \tag{26}$$

and $\beta_1,\beta_2,\beta_3,..........$ are the positive roots of transcendental equation

$$\frac{J_0(\beta_m a)}{J_0'(\beta_m b)}-\frac{Y_0(\beta_m a)}{Y_0'(\beta_m b)}=0 \tag{27}$$

This transform satisfies the relation

$$H\left[\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}\right]=-\beta_m^2\ \bar{\theta}(\beta_m,z,t) \tag{28}$$

Secondly, the finite Fourier transform over the variable z and its inverse transform for the Hankel-transformed function $\bar{\theta}(\beta_m,z,t)$, defined in[16] as

$$\bar{\bar{\theta}}(\beta_m,\eta_p,t)=\int_{z'=-h/2}^{h/2}K(\eta_p,z')\,\bar{\theta}(\beta_m,z',t)dz' \tag{29}$$

$$\bar{\theta}(\beta_m,z,t)=\sum_{p=1}^{\infty}K(\eta_p,z)\bar{\bar{\theta}}(\beta_m,\eta_p,t) \tag{30}$$

Where

$$K(\eta_p,z)=\sqrt{\frac{2}{\pi}}\ \sin(\eta_p z) \tag{31}$$

and $\eta_1,\eta_2,\eta_3,..........$ are the positive roots of transcendental equation

$$\sin(\eta_p\, z/2)=0\ ,\qquad p=1,2,3,........$$

Applying Fourier, Hankel and Laplace transform and their inversions to equation (18) and making use of the transformed boundary and initial conditions (19)-(23), one get

$$\theta(r,z,t)=\frac{Q_0}{\lambda}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)K_0(\beta_m,r)$$
$$\times\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,z/2)\int_{r'=a}^{b}r'\ K_0(\beta_m,r')\ f(r')dr'\right]\left[E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha\right] \tag{32}$$

Where

$$L^{-1}\left[\frac{1}{s^{\alpha}+k(\beta_m^2+\eta_p^2)}\right]=\left[E_{\alpha}(-k(\beta_m^2+\eta_p^2)t^{\alpha}\right]$$

Here $E_{\alpha}(.)$ represents the Mittag-Leffler function.

For the cooling process the temperature change $\theta'(r,z,t)$ satisfies the equations

$$\frac{\partial^2\theta'}{\partial r^2}+\frac{1}{r}\frac{\partial\theta'}{\partial r}+\frac{\partial^2\theta'}{\partial z^2}=\frac{1}{k}\frac{\partial^{\alpha}\theta'}{\partial t^{\alpha}} \tag{33}$$

$$\left.\frac{\partial\theta'}{\partial r}\right|_{r=a}=0 \tag{34}$$

$$\left.\frac{\partial\theta'}{\partial r}\right|_{r=b}=0 \tag{35}$$

$$\left.\frac{\partial\theta'}{\partial z}\right|_{z=h/2}=0 \tag{36}$$

$$\left.\frac{\partial\theta'}{\partial z}\right|_{z=-h/2}=0 \tag{37}$$

$$\theta'=\theta(r,z,t_0) \quad at \quad t=t_0 \tag{38}$$

Repeating the above process, we can find out that the temperature solution θ' for the cooling process as

$$\theta'(r,z,t)=\frac{Q_0}{\lambda}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)K_0(\beta_m,r)$$
$$\times\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,z/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_{\alpha}(-k(\beta_m^2+\eta_p^2)t^{\alpha}\right]-\theta(r,z,t-t_0) \tag{39}$$

Determination of Displacement Potential and Thermal Stresses

Using equation (32) and (1), we get displacement potential function as follows

$$U=\frac{Q_0}{\lambda}(1+v)a_t\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)\frac{1}{\beta_m^2}K_0(\beta_m,r)$$
$$\times\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,z/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_{\alpha}(-k(\beta_m^2+\eta_p^2)t^{\alpha}\right] \tag{40}$$

Using equations (40) in (4), we obtain radial and angular stresses as follows

$$\sigma_{rr}=\frac{Q_0}{\lambda}2(1+v)a_t\mu\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)\frac{1}{r\beta_m^2}K_1(\beta_m,r)$$
$$\times\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,z/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha\right] \tag{41}$$

$$\sigma_{\theta\theta}=\frac{Q_0}{\lambda}2(1+v)a_t\mu\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}K(\eta_p,z)\frac{1}{\beta_m^2}\left(\beta_m K_0(\beta_m,r)-\frac{K_1(\beta_m,r)}{r}\right)$$
$$\times\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,z/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha\right] \tag{42}$$

Where

$$K_1(\beta_m,r)=\frac{\partial}{\partial r}K_0(\beta_m,r)$$

Determination of the transverse Deflection $w(r,t)$

Assume the solution of Eq. (8) satisfy condition (12) as

$$w(r,t)=\sum_{m=1}^{\infty}C_m(t)\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)}-\frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \tag{43}$$

Here $\beta_1,\beta_2,\beta_3,..........$ are the positive roots of transcendental equation

$$\frac{J_0'(\beta a)}{J_0'(\beta b)}-\frac{Y_0'(\beta a)}{Y_0'(\beta b)}=0$$

It can be easily shown that

$$\frac{\partial w}{\partial r}=\sum_{m=1}^{\infty}C_m(t)\left[\frac{J_0'(\beta a)}{J_0'(\beta b)}-\frac{Y_0'(\beta a)}{Y_0'(\beta b)}\right] \tag{44}$$

$$\frac{\partial w}{\partial r}=0 \quad at \quad r=a \quad and \quad r=b$$

Hence the solution (44) satisfies the equation (8).

Now

$$\nabla^2\nabla^2 w=\left(\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}\right)^2\sum_{m=1}^{\infty}C_m(t)\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)}-\frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \tag{45}$$

On simplifying equation (45), one obtains

$$C_m(t)=\sqrt{\frac{2}{\pi}}\,2\,a_t\,E\frac{1}{(1-v)D}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin(\eta_p,h/2)\frac{1}{\beta_m^2}$$
$$\times\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m,b)Y_0'(\beta_m,b)}{\sqrt{1-\frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}}}\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,h/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha\right] \tag{46}$$

Using (46) in (45), one obtain the expression for $w(r,t)$ as

$$w(r,t)=\sqrt{\frac{2}{\pi}}\,2a_t\,E\frac{1}{(1-v)D}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin(\eta_p,h/2)\frac{1}{\beta_m}$$
$$\times\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m,b)Y_0'(\beta_m,b)}{\sqrt{1-\frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}}}\left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)}-\frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right]\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,h/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha\right] \tag{47}$$

Substituting equating (47) in (14)–(16), the expressions for resultant moments and shearing forces obtained as

$$M_{rr}=\sqrt{\frac{2}{\pi}}\,2a_t\,E\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin(\eta_p,h/2)\frac{1}{r\beta_m}$$
$$\times\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m,b)Y_0'(\beta_m,b)}{\sqrt{1-\frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}}}\left[\frac{J_0'(\beta_m r)}{J_0'(\beta_m b)}-\frac{Y_0'(\beta_m r)}{Y_0'(\beta_m b)}\right]\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,h/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha\right] \tag{48}$$

$$M_{\theta\theta}=\sqrt{\frac{2}{\pi}}\,2a_t\,E\frac{1}{(1-v)D}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin(\eta_p,h/2)\frac{1}{r\beta_m}$$
$$\times\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m,b)Y_0'(\beta_m,b)}{\sqrt{1-\frac{J_0'^2(\beta_m b)}{J_0'^2(\beta_m a)}}}\left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)}-\frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}-\frac{1}{r\beta_m\left(\frac{J_0'(\beta_m r)}{J_0'(\beta_m b)}-\frac{Y_0'(\beta_m r)}{Y_0'(\beta_m b)}\right)}\right] \tag{49}$$
$$\times\left[\frac{\lambda}{k}\frac{h}{2}K(\eta_p,h/2)\int_{r'=a}^{b}r'\,K_0(\beta_m,r')\,f(r')dr'\right]\left[E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha\right]$$

And $Q_r=Q_\theta=M_{r\theta}=0$ (50)

CONCLUSIONS

In this work we discussed fractional order theory of thermoelasticity for a thin Hollow Circular Disk where the initial temperature of the disk and the temperature of the surrounding medium are assumed to be same, which is kept constant. The upper surface of disk is subjected to axisymmetric and partially distributed heat supply, while the lower surface is kept thermally insulated. The associated thermal stresses and Deflection are obtained by using the displacement potential function with the use of Integral transform technique. The infinite wave propagation in terms of heat energy is observed due to the parabolic nature of heat conduction equation.

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