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⋆-closure Operators on Čech Closure Space

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ABSTRACT

Some results are introduced by using Čech closure operators and ⋆-closure operator. Also it defines ⋆-closure operator on Čech closure space as the generalized ⋆-closure operator and study some of its properties.

Keywords: closure operator, ⋆-closure operator , $\star_g$-closure operator.

INTRODUCTION

Čech closure space were introduced by Čech[3]. In Čech′s approach the operator satisfies idempotent condition among Kuratowski axioms. This condition need not hold in some cases. When this condition is also true, the operator becomes topological closure operator. Thus the concept of closure space is the generalization of topological space. Pratibha Bhat and Ananga Kumar Das[5] introduced the concept of ⋆-closure operator and studied some of its properties. In the topological literature several forms of closed sets exist which has been utilized by researchers to study various topological properties. Closure of a set need not be a closed set, which depends on the pre-defined closure operator. As the tittle indicates, this paper discussed Čech closure operators and ⋆-closure operators on a set X and introduced the ⋆-closure operators on Čech closure space and studied some of its properties. Also it studies the properties of set of all ⋆-closure operators on a set.

1. PRELIMINARIES

Definition 1.1 A Čech closure operator V on a set X is a function $V: \wp(X) \to \wp(X)$ such that,
(i) $V(\phi) = \phi$
(ii) $A \subseteq V(A)$ for all $A \in \wp(X)$
(iii) $V(A \cup B) = V(A) \cup V(B)$ for all $A, B \in \wp(X)$, where $\wp(X)$ denotes the power set of X.

The pair (X, V) is called a Čech closure space. For brevity, we call V a closure operator on X and the pair (X, V) a closure space.

A subset A in a closure space (X, V) is said to be open if $V(X - A) = X - A$. The set of all open sets in (X, V) form a topology on X, called the topology associated with V. The closed sets of (X, V) are exactly the fixed points of V. By definition itself, $A \subseteq V(A)$ for all $A \subseteq X$. Thus a subset A of X is closed, if $A \supseteq V(A)$. On other hand, to every topology τ on X, we can associate a closure operator $'cl'$ on X called the Kuratowski closure operator on X by defining $'cl(A)'$ as the smallest closed set in (X, τ) containing A. The kuratowski closure operator $'cl'$ satisfies all the three conditions of the Čech closure operator and an additional condition $cl(cl(A)) = cl(A)$ for every $A \in \wp(X)$.

Remark 1.2 A closure operator on a set is not uniquely determined by its associated topology. That is, two different closure operators on a set can have the same topology as the associated topology.

Remark 1.3 If a closure operator V on a set X satisfies the additional condition $V(V(A)) = V(A)$ for every $A \in \wp(X)$ (we say that V is idempotent), then the Kuratowski closure operator of the associated topology of V is V itself and in this case, V is also said to be topological. Note that a closure operator need not be the closure operator associated with the topology associated with it.

Remark 1.4 In a closure space, the closure of a set need not be closed. For example, let $X = \{x, y, z\}$ and V be the closure operator defined by $V(\{x\}) = \{x, y\}$, $V(\{y\}) = \{y, z\}$, $V(\{z\}) = \{z, x\}$. (Note that a closure operator on a finite set is determined by the closures of singleton sets). Then the closure of $\{x\}$ is not closed, since $V(V(\{x\})) = V(\{x, y\}) = \{x, y, z\} \neq V(\{x\})$.

Theorem 1.5 There cannot exist two different closure operators on a set with the discrete topology as the associated topology.

Definition 1.6 With any closure operator V on a set X, there is associated the interior operator int_V, which is a mapping from $\wp(X) \to \wp(X)$ such that for each $A \subseteq X, int_V(A) = X - V(X - A)$. The set $int_V(A)$ is called the interior of A in (X, V) or the $V-$interior of A in X.

From the definition of a closure operator and interior operator,we immediately obtain the following assertion: In any space the following three conditions are fulfilled,

i) $int(X) = X$
ii) $int(A) \subseteq A$ for all $A \subseteq X$.
iii) $int(A \cap B) = int(A) \cap int(B)$ for all $A, B \subseteq X$.

If V is a closure operator on X, and int_V is the corresponding interior operator, then $int_V(A) = X - V(X - A)$ and $V(A) = X - int_V(X - A)$. Thus the closure operator on a set is uniquely determined by the interior operator and the interior operator on a set is uniquely determined by the closure operator.

Definition 1.7 Let X be any topological space, then an operator $\star:\wp(X) \to \wp(X)$ defined by $\star(A)=cl(A^0)$ is known as $\star$-closure operator. Where A^0 is the interior of A.
It can be easily verified that $\star$-closure operator satisfies the following axioms:
(1) Monotone i.e. $A \subset B \Longrightarrow \star(A) \subset \star(B)$ for every $A, B \subseteq X$.
(2) Idempotence i.e. $\star(\star(A)) = \star(A)$ for every $A \subseteq X$.
(3) Join homomorphism i.e. $\star(A \cup B) = \star(A) \cup \star(B)$ for every $A, B \subseteq X$.

2. SOME PROPERTIES OF $\star$-CLOSURE OPERATOR

Remark 2.1 For every topological space there exist a $\star$-closure operator.

Proof: It is obvious from the definition.

Remark 2.2 $\star$-closure operator need not satisfy meet homomorphism.

Example 2.3 Let X be the real line with usual topology and let A = [0, 1] and B= [1, 2]. Then $\star(A) \cap \star(B) = \{1\}$ and $\star(A \cap B) = \phi$. Thus $\star(A \cap B) \neq \star(A) \cap \star(B)$.

Remark 2.4 $\star(A^C)$ need not be equal to $\left(\star\,(A)\right)^C$, where A^C is the complement of A with respect to X.

Example 2.5 Let $X=\{x, y, z\}$ and V be the closure operator mentioned in the Remark 1.6 and let $A = \{x\}$, $\star(A^C) = \star\,(\{y, z\}) = X$. But $\left(\star\,(A)\right)^C = \left(\star\,(x)\right)^C = (\{x, y\})^C = z$

Remark 2.6 $\star$-closure operator need not satisfy inflationary axiom .

Example 2.7 Let $X= \{a, b, c\}$ and $\tau_X = \{\phi, X, \{a\}, \{b, c\}\}$. For $A= \{a, b\}$,$\star(A)= \{a\}$. Thus $\star(A)$ does not contain A.
From the above example it is clear that $\star\,(A)$ is non-inflationary, $\star\,(A)$ satisfies any one of the following cases:
(case I) $A= \star(A)$.
(case II) $A \subseteq \star(A)$.
(case III) $\star(A) \subseteq A$
For any topological space if a set satisfies case I, then the set is regularly closed, if it satisfies case II , then the set is semi-open and sets satisfying case III are known as pre-closed sets.

3. $\star$-CLOSURE OPERATOR ASSOCIATED WITH CLOSURE OPERATOR

For any closure space, there exist a $\star$-closure operator. Since for every closure space, there exist a topology associated with closure operator.

Lemma 3.1 Two different closure operators on a set can have the same $\star$-closure operator.

Proof: From the Remark 1.4, two different closure operators on a set can have the same topology as the associated topology. This case leads to same $\star$-closure operator exist for different closure operators.

Theorem 3.2 Let (X, V_1) and (X, V_2) be two closure spaces and have same $\star$-closure operator. If V_1 and V_2 are idempotent, then $V_1 = V_2$.

Proof: (X, V_1) and (X, V_2) have same $\star$-closure operator means that the associated topology for both closure space is same and V_1 and V_2 are idempotent. That implies $V_1 = V_2 = cl$, Kuratowski closure operator associated with this topology.

4. SET OF $\star$-CLOSURE OPERATORS

Definition 4.1 Let $\star_1$ and $\star_2$ be two $\star$-closure operators on a set X. Then $\star_1$ is said to be "coarser than" $\star_2$ ($\star_2$ is said to be finer than $\star_1$) if $\star_1(A) \subseteq \star_2(A)$ for all $A \epsilon \wp(X)$. In this case we write $\star_1 \leq \star_2$.

Theorem 4.2 The set of all $\star$-closure operators on a set with the partial ordering $\leq$ defined above is a poset (partially ordered set).

Example 4.3 Let $X = \{x, y\}$. Consider all possible topologies of a set X and its corresponding $\star$-closure operators. Let τ_i be the topology on X and $\star_i$ be the $\star$-closure operator associated with τ_i, where $i = 1,2,3,4$.

1) $\tau_1 = \{\phi, X\}$
   $\star_1(\phi)=\phi$ $\star_1(y)=\phi$
   $\star_1(x)=\phi$ $\star_1(X)=X$
2) $\tau_2 = \{\phi, x, X\}$,
   $\star_2(\phi)=\phi$ $\star_2(y)=\phi$
   $\star_2(x)=X$ $\star_2(X)=X$
3) $\tau_3 = \{\phi, y, X\}$,
   $\star_3(\phi)=\phi$ $\star_3(y)=X$
   $\star_3(x)=\phi$ $\star_3(X)=X$
4) $\tau_4 = \{\phi, x, y, X\}$,
   $\star_4(\phi)=\phi$ $\star_4(y)=y$
   $\star_4(x)=x$ $\star_4(X)=X$

Theorem 4.4 There cannot exist two any closure operators on a set X(X having more than one element) with the operator $\star$ defined below as the associated $\star$-closure operator.

An operator $\star:\wp(X) \rightarrow \wp(X)$ defined by $\star(A) = \begin{cases} \phi & if\ A = \phi \\ X & otherwise \end{cases}$

Theorem 4.5 Let $ST(X)$ be the set of all topologies on a set X and $SC_\star(X)$ be the set of $\star$-closure operators on a set X. The function $f: ST(X) \longrightarrow SC_\star(X)$ defined by $f(\tau) = \star_\tau$, $\tau \in ST(X)$ and $\star_\tau$ be the corresponding $\star$-closure operator, is onto.

Proof: For every $\star$-closure operator there exist a topology .

Theorem 4.6 Let $SC(X)$ be the set of closure operators on a set X. . The function $f: SC(X) \longrightarrow SC_{\star}(X)$ defined by $f(V) = \star_V$, $V \in SC(X)$and $\star_V$ be the corresponding $\star$-closure operator, is onto.

Proof: Using Lemma 3.1, we say that f is not one-one. For every element ($\star$-closure operator) in $SC_{\star}(X)$, there exist atleast one closure operator(for example Kuratowski closure operator) which induces that element. That implies f onto.

Remark 4.7 Let $\star_1$ and $\star_2$ be two $\star$-closure operators on a set X. If we define $\star: \wp(X) \to \wp(X)$ by $\star(A) = \star_1(A) \cup \star_2(A)$ for $A \in \wp(X)$, then $\star$ is a $\star$-closure operator on (X,V).

Remark 4.8 Let $\star_1$ and $\star_2$ be two $\star$-closure operators on a set X. If $\star: \wp(X) \to \wp(X)$defined by $\star(A) = \star_1(A) \cap \star_2(A)$ for $A \in \wp(X)$, then $\star$ need not be a $\star$-closure operator on(X, V), as $\star$ need not satisfy the third condition for $\star$-closure operator.

Example 4.9 Let (X, τ_1) and (X, τ_2) be two topological spaces. Where $X = \{x, y, z\}$,
τ_1 be the topology , $\tau_1 = \{\phi, y, z, \{y, z\}, X\}$,
$\star_1$ be corresponding (w.r.t τ_1) $\star$-closure operator, $\star_1(\{x\}) = \phi$, $\star_1(\{y\}) = \{x, y\}$, $\star_1(\{z\}) = \{x, z\}$
and
τ_2 be the topology, $\tau_2 = \{\phi, x, z, \{x, z\}, X\}$,
$\star_2$ be corresponding (w.r.t τ_2) $\star$-closure operator, $\star_2(\{x\}) = \{x, y\}$, $\star_2(\{y\}) = \phi$, $\star_2(\{z\}) = \{y, z\}$
Then $\star: \wp(X) \to \wp(X)$ defined by $\star(A) = \star_1(A) \cap \star_2(A)$ for $A \in \wp(X)$,
$\star(\{x, y\}) = \star_1(\{x, y\}) \cap \star_2(\{x, y\}) = \{x, y\} \cap \{x, y\} = \{x, y\}$ ①
$\star(\{x\}) = \star_1(\{x\}) \cap \star_2(\{x\}) = \phi \cap \{x, y\} = \phi$and
$\star(\{y\}) = \star_1(\{y\}) \cap \star_2(\{y\}) = \{x, y\} \cap \phi = \phi$ ②
Equations ① and② implies $\star(\{x, y\}) \neq \star(\{x\}) \cup \star(\{y\})$. That implies $\star$ defined above is not a $\star$-closure operator.

5. GENERALIZED $\star$-CLOSURE OPERATOR

Definition 5.1 Let (X, V) be a closure space, an operator $\star_g: \wp(X) \to \wp(X)$ defined by $\star_g(A) = V(int_V(A))$ is known as generalized $\star$-closure operator or $\star_g$-closure operator.

Remark 5.2 Any set can have atleast one $\star_g$-closure operator with the same associated $\star$-closure operator.

Theorem 5.3 Let (X, V) be a closure space , $\star$-closure operator and $\star_g$-closure operator are same if V is idempotent.

Proof: If V is idempotent, then V is topological. That implies $\star$-closure operator and $\star_g$-closure operator are same if V is idempotent.

Example 5.4 Let (X,V) be the closure space , where $X=\{x,y,z\}$ and V is defined by $V(\{x\})=\{x,y\}, V(\{y\})=\{y,z\}, \quad V(\{z\})=\{z,x\}$
And let τ be the topology associated with V, τ = Set of all open sets in X= $\{\phi$, X$\}$, indiscrete topology. Then $\star$-closure operator $\star:\wp(X)\rightarrow\wp(X)$ and $\star_g$-closure operator $\star_g:\wp\ (X)\rightarrow\wp(X)$ is given below.
$\star(\{x\})=\phi, \quad \star(\{y\})=\phi, \quad \star(\{z\})=\phi, \quad \star(\{x,y\})=\phi, \quad \star(\{y,z\})=\phi, \quad \star(\{x,z\})=\phi, \quad \star(X)=X$
and
$\star_g(\{x\})=\phi, \quad \star_g(\{y\})=\phi, \quad \star_g(\{z\})=\phi, \quad \star_g(\{x,y\})=y, \star_g(\{y,z\})=z, \quad \star_g(\{x,z\})=x, \star_g(X)=X$
Here $\star$-closure operator not equal to $\star_g$-closure operator.

Remark 5.5 $\star_g$-closure operator need not satisfy join homomorphism.

Proof: Let(X,V) be a closure space and let A and $B\subseteq X$.
$\star_g\ (A)\cup\star_g\ (B)=V\big(int_V(A)\big)\cup V\big(int_V(B)\big)=V(int_V(A)\cup int_V(B))$, Since $int_V(A)\cup int_V(B)$ need not be equal to $int_V(A\cup B)$. This implies $\star_g\ (A\cup B)$ need not be equal to $\star_g\ (A)\cup\star_g\ (B)$.

Remark 5.6 $\star_g$-closure operator need not satisfy meet homomorphism.

Proof: Let(X,V) be a closure space and let A and $B\subseteq X$. $\star_g\ (A\cap B)=V\big(int_V(A\cap B)\big)=V(int_V(A)\cap int_V(B))$ need not be equal to $V\big(int_V(A)\big)\cap V\big(int_V(B)\big)=\ \star_g\ (A)\cap\star_g\ (B)$. Since closure operator need not satisty meet homomorphism.

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