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Fixed Point Theorem Using for Weakly Compatible Mappings in E–Fuzzy Metric Spaces

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ABSTRACT

In this paper, we prove common fixed point theorem in E–fuzzy metric space using weakly compatible mappings.

Keywords: Fuzzy metric space, G–metric space, weakly compatible, self mappings, E–fuzzy metric space.

1. INTRODUCTION

The concept of fuzzy sets was initially introduced by Zadeh[2]. It is a new way to represent distinctness in our daily life. Subsequently, it was developed by many authors and used in various fields. Kramosil and Michalek[7] has introduced fuzzy metric space to use this concept in Topology and Analysis. George and Veeramani modified the concept of fuzzy metric space. A. Al–Thagafi and Naseer Shahzad introduced the concept of occasionally weakly compatible maps. G. Jungck and B.E. Rhoades[6] also proved fixed point theorem for occasionally weakly compatible mappings.

In 2006, Mustafa Z. ad B. Sims[4] presented a definition of G–metric space. K P Sukanya and J. Sr. Magic generalized E–fuzzy metric space. In this paper, we prove common fixed point theorem for six mappings satisfying weakly compatible condition in this space which generalized the results of[8].

2. PRELIMINARY NOTES

Definition 2.1: A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2: A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t–norm if * satisfies the following condition

- * is commutative and associative
- * is continuous
- $a * 1 = a$ for all $a \in [0, 1]$
- $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3: A 3–tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t–norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following condition for all $x, y, z \in X$ and $s, t > 0$.

- $M(x, y, t) > 0$
- $M(x, y, t) = 1$, if and only if $x = y$
- $M(x, y, t) = M(y, x, t)$
- $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- $M(x, y, \bullet) : (0, \infty) \rightarrow (0, 1]$ is continuous M(x, y, t) denotes the degree of nearness between x and y with respect to t.

Definition 2.4: Let X be a non empty set and Let $G : X \times X \times X \rightarrow [0, \infty)$ be a function satisfying the following

- $G(x, y, z) = 0$ if $x = y = z$
- $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$
- $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$
- $G(x, y, z) = G(P\{x, y, z\})$ (symmetry) where P is a permutation function.
- $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (Rectangle inequality)
  Then the function is called generalized metric, or more specifically a G–metric on X and the pair (X, G) is a G–metric space.

Definition 2.5: Let X be a set. Let f and g be self maps on X. A point y in X is called a coincidence point of f and g if and only if fy = gy. In this case w = fy = gy is called point of coincidence of f and g.

Definition 2.6: Let X be a set. A pair of self mappings (f, g) on X is said to be weakly compatible if they commute at the coincidence points, that is, if ft = gt for some $t \in X$, then fgt = gft.

Definition 2.7: A 3–tuple $(X \times E, *)$ is said to be on E–fuzzy metric space if X is an arbitrary set, * is a continuous t–norm and E is a fuzzy set on $X^3 \times (0, \infty)$ which satisfies the following conditions for each $x, y, z, a \in X$ and $t, s > 0$.

- $E(x, x, y, t) > 0$ and $E(x, x, y, t) \geq E(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq y$.

- E(x, y, z, t) = 1, for all t > 0 if and only if x = y = z.
- E(x, y, z, t) = E(p(x, y, z), t) (symmetry), where p is a permutation function.
- E(x, a, z, t) * E(a, y, z, s) ≤ E(x, y, z, t + s)
- E(x, y, z, •) : (0, ∞) → [0, 1] is continuous.

E–fuzzy metric space gives a generalization of fuzzy metric space.

Example: Consider a non–empty set X and a G–metric G on X. The t–norm is a * b = ab for all a, b ∈ [0, 1]. For each t > 0, $E(x, y, z, t) = \left[\exp\left(\frac{G(x,y,z)}{t}\right)\right]^{-1}$. E satisfies all the conditions in the above definition and hence (X, E, *) is a E–fuzzy metric space.

Lemma: In an E–fuzzy metric space (X, E, *), E(x, y, z, t) is non–decreasing with respect to t for all x, y, z ∈ X.

Proof: By taking a = x and z = x in the condition E(x, a, z, t) * E(a, y, z, s) ≤ E(x, y, z, t + s) we get E(x, x, y, t) ≤ E(x, x, y, t + s).
If possible, Let E(x, y, z, t) > E(x, y, z, t + s).
Again if we put z = x in the above conditions we get a contradiction. Hence the result.

3. FIXED POINT THEOREM

Theorem 3.1: Let (X, E, *) is a complete E–fuzzy metric space. A, B, C, S, T and U are self mapping of X. The pairs of mappings {A, S}, {B, T} and {C, U} are weakly compatible. Also S(X) ⊂ B(X), T(X) ⊂ C(X) and U(X) ⊂ A(X). Also A(X) is complete. If there exist k ∈ (0, 1) such that

$$E(Sx, Ty, Uz, kt) \geq \text{Min}\left\{\begin{matrix} E(Ax, By, Cz, t) \\ E(Sx, Ax, By, t) \\ E(Sx, By, Cz, t) \\ E(Uz, Cz, Ax, t) \\ E(Uz, Ax, By, t) \end{matrix}\right\}$$

For all x, y, z ∈ X. Then these six mapping A, B, C, S, T and U have a unique common fixed point.

Proof: Let $x_0 \in X$. Let $S(x_0) = y_1$. Also since S(X) ⊂ B(X), T(X) ⊂ C(X), U(X) ⊂ A(X), by iteration we can define sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{3n+1} = Sx_{3n} = Bx_{3n+1},\ y_{3n+2} = Tx_{3n+1} = Cx_{3n+2},\ y_{3n+3} = Ux_{3n+2} = Ax_{3n+3}$$

$$E(Sx, Ty, Uz, kt) \geq Min \begin{Bmatrix} E(Ax, By, Cz, t) \\ E(Sx, Ax, By, t) \\ E(Sx, By, Cz, t) \\ E(Uz, Cz, Ax, t) \\ E(Uz, Ax, By, t) \end{Bmatrix}$$

$$E(Sx_{3n}, Tx_{3n+1}, Ux_{3n+2}, kt) \geq min \begin{Bmatrix} E(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}, t) \\ E(Sx_{3n}, Ax_{3n}, Bx_{3n+1}, t) \\ E(Sx_{3n}, Bx_{3n+1}, Cx_{3n+2}, t) \\ E(Ux_{3n+2}, Cx_{3n+2}, Ax_{3n}, t) \\ E(Ux_{3n+2}, Ax_{3n}, Bx_{3n+1}, t) \end{Bmatrix}$$

$$E(y_{3n+1}, y_{3n+2}, y_{3n+3}, kt) \geq Min \begin{Bmatrix} E(y_{3n}, y_{3n+1}, y_{3n+2}, t) \\ E(y_{3n+1}, y_{3n}, y_{3n+1}, t) \\ E(y_{3n+1}, y_{3n+1}, y_{3n+2}, t) \\ E(y_{3n+3}, y_{3n+2}, y_{3n}, t) \\ E(y_{3n+2}, y_{3n}, y_{3n+1}, t) \end{Bmatrix}$$

Similarly, we have $E(y_n, y_{n+1}, y_{n+2}, kt) \geq E(y_{n-1}, y_n, y_{n+1}, t)$.
$\{y_n\}$ is Cauchy sequence and also X is complete, hence we can find z in X such that $y_n \to z$.
So the subsequences (y_{3n}), (y_{3n+1}), (y_{3n+2}) are also convergent.
i.e. $Lim\ Bx_{3n+1} = Lim\ Sx_{3n} = Lim\ Cx_{3n+2} = Lim\ Tx_{3n+1} = Lim\ Ax_{3n+3} = Lim\ Ux_{3n+2} = z$.
A(x) is complete.
So, there exists w in S such that Aw = z.
Next we claim that Sw = z.

$$E(Sw, Tx_{3n+1}, Ux_{3n+2}, kt) \geq Min \begin{Bmatrix} E(Aw, Bx_{3n+1}, Cx_{3n+2}, t) \\ E(Sw, Aw, Bx_{3n+1}, t) \\ E(Sw, Bx_{3n+1}, Cx_{3n+2}, t) \\ E(Ux_{3n+2}, Cx_{3n+2}, Aw, t) \\ E(Ux_{3n+2}, Aw, Bx_{3n+1}, t) \end{Bmatrix}$$

$$E(Sw, y_{3n+2}, y_{3n+3}, kt) \geq Min \begin{Bmatrix} E(z, y_{3n+1}, y_{3n+2}, t) \\ E(Sw, z, y_{3n+1}, t) \\ E(Sw, y_{3n+1}, y_{3n+2}, t) \\ E(y_{3n+2}, y_{3n+2}, z, t) \\ E(y_{3n+2}, z, y_{3n+1}, t) \end{Bmatrix}$$

Taking limit $n \to \infty$

$$E(Sw, z, z, kt) \geq \text{Min}\begin{Bmatrix} E(z, z, z, t) \\ E(Sw, z, z, t) \\ E(Sw, z, z, t) \\ E(z, z, z, t) \\ E(z, z, z, t) \end{Bmatrix}$$

E(Sw, z, z, kt) $\geq$ E(Sw, z, z, t)
i.e. E(Sw, z, z, kt) = 1

Therefore, Sw = z = Aw. Hence, w is a coincidence point of S and A.
$S(X) \subset B(X)$, i.e. $z \in S(X) \subset B(x)$, thus we have $t \in X$ such that Bt = z.

$$E(Sx_{3n}, Tt, Ux_{3n+2}, kt) \geq \text{Min}\begin{Bmatrix} E(Ax_{3n}, Bt, Cx_{3n+2}, t) \\ E(Sx_{3n}, Ax_{3n}, Bt, t) \\ E(Sx_{3n}, Bt, Cx_{3n+2}, t) \\ E(Ux_{3n+2}, Cx_{3n+2}, Ax_{3n}, t) \\ E(Ux_{3n+2}, Ax_{3n}, Bt, t) \end{Bmatrix}$$

$$E(y_{3n+1}, Tt, y_{3n+3}, kt) \geq \text{Min}\begin{Bmatrix} E(y_{3n}, z, y_{3n+2}, t) \\ E(y_{3n+1}, y_{3n}, z, t) \\ E(y_{3n+1}, z, y_{3n+2}, t) \\ E(y_{3n+3}, y_{3n+2}, y_{3n}, t) \\ E(y_{3n+3}, y_{3n}, z, t) \end{Bmatrix}$$

Taking limit $n \to \infty$

$$E(z, Tt, z, kt) \geq \text{Min}\begin{Bmatrix} E(z, z, z, t) \\ E(z, z, z, t) \\ E(z, z, z, t) \\ E(z, z, z, t) \\ E(z, z, z, t) \end{Bmatrix}$$

i.e. E(z, Tt, z, kt) = 1

Then Tt = z = Bt, thus t is a coincidence point of B and T.
Now $T(X) \subset C(X)$ and $z = T(t) \in T(X) \subset C(X)$.

Hence, there exists a point $v \in X$ such that Cv = z.

$$E(Sx_{3n}, Tx_{3n+1}, Uv, kt) \geq Min \begin{Bmatrix} E(Ax_{3n}, Bx_{3n+1}, Cv, t) \\ E(Sx_{3n}, Ax_{3n}, Bx_{3n+1}, t) \\ E(Sx_{3n}, Bx_{3n+1}, Cv, t) \\ E(Uv, Cv, Ax_{3n}, t) \\ E(Uv, Ax_{3n}, Bx_{3n+1}, t) \end{Bmatrix}$$

$$E(y_{3n+1}, y_{3n+2}, Uv, kt) \geq Min \begin{Bmatrix} E(y_{3n}, y_{3n+1}, z, t) \\ E(y_{3n+1}, y_{3n}, y_{3n+1}, t) \\ E(y_{3n+1}, y_{3n+1}, z, t) \\ E(Uv, z, y_{3n}, t) \\ E(Uv, y_{3n+1}, y_{3n+2}, t) \end{Bmatrix}$$

Taking limit $n \to \infty$,

$$E(z, z, Uv, kt) \geq Min \begin{Bmatrix} E(z, z, z, t) \\ E(z, z, z, t) \\ E(z, z, z, t) \\ E(Uv, z, z, t) \\ E(Uv, z, z, t) \end{Bmatrix}$$

$E(z, z, Uv, kt) \geq E(Uv, z, z, t)$

Hence Uv = z. We have Uv = Cv = z.

Since {A, S}, {B, T}, {C, U} are weakly compatible, they commute at coincidence points.

We have Sw = z = Aw, then ASw = SAw i.e. Az = Sz

Also Tt = z = Bt. Then BTt = TBt i.e. Bz = Tz.

Since {C, U} is weakly compatible, similarly we get Cz = Uz.

$$E(Sz, Tx_{3n+1}, Uz, kt) \geq Min \begin{Bmatrix} E(Az, Bx_{3n+1}, Cz, t) \\ E(Sz, Az, Bx_{3n+1}, t) \\ E(Sz, Bx_{3n+1}, Cz, t) \\ E(Uz, Cz, Az, t) \\ E(Uz, Az, Bx_{3n+1}, t) \end{Bmatrix}$$

Taking limit $n \to \infty$

$$E(Sz, z, Uz, kt) \geq Min \begin{Bmatrix} E(Sz, z, Uz, t) \\ E(Sz, Sz, z, t) \\ E(Sz, z, Uz, t) \\ E(Uz, Uz, Sz, t) \\ E(Uz, Sz, z, t) \end{Bmatrix}$$

E(Sz, z, Uz, kt) ≥ E(Sz, z, Uz, t). Thus, we have Sz = z = Uz.
Hence Az = Sz = Cz = Uz = z.

$$E(z, Tz, z, kt) = E(Sz, Tz, Uz, kt) \geq Min \begin{Bmatrix} E(Az, Bz, Cz, t) \\ E(Sz, Az, Bz, t) \\ E(Sz, Bz, Cz, t) \\ E(Uz, Cz, Az, t) \\ E(Uz, Az, Bz, t) \end{Bmatrix}$$

$$\geq Min \begin{Bmatrix} E(z, Tz, z, t) \\ E(z, z, Tz, t) \\ E(z, Tz, z, t) \\ E(z, z, z, t) \\ E(z, z, Tz, t) \end{Bmatrix}$$

E(z, Tz, z, kt) ≥ E(z, Tz, z, t). Hence Tz = Bz = z.
Thus Az =Sz = Bz = Tz = Cz = Uz = z.

Then z is a common fixed point of A, B, C, S, T and U. To prove Uniqueness, Let z' be another common fixed point of A, B, C, S, T and U.

$$E(z, z', z, kt) = E(Sz, Tz', Uz, kt) \geq Min \begin{Bmatrix} E(Az, Bz', Cz, t) \\ E(Sz, Az, Bz', t) \\ E(Sz, Bz', Cz, t) \\ E(Uz, Cz, Az, t) \\ E(Uz, Az, Bz', t) \end{Bmatrix}$$

$$E(z, z', z, kt) \geq Min \begin{Bmatrix} E(z, z', z, t) \\ E(z, z, z', t) \\ E(z, z', z, t) \\ E(z, z, z, t) \\ E(z, z, z', t) \end{Bmatrix}$$

$$E(z, z', z, kt) \geq E(z, z', z, t)$$

Hence z' = z.

4. CONCLUSION

Fixed point theory has been studied in different spaces. In this paper we prove common fixed point theorem for six mappings satisfying weakly compatible condition in E-Fuzzy metric space. Our results presented in this paper improve some known results in fuzzy metric space[8].

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