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Fixed Point Theorem for ϕ-weak Contraction Mappings

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ABSTRACT

In this paper, we present some common fixed point theorem for generalized ϕ-weak contraction mappings. Also, we discuss the existence and uniqueness of common fixed points for single valued mappings satisfying the notion of weak compatibility in a complete metric space.

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1. INTRODUCTION

Fixed point theory is an extremely active area of research due to its applications in multiple fields. It analyse the results which state that under certain conditions a self map on a set admits a fixed point. Among all the results in fixed point theory 'Banach Contraction Principle' in metric fixed point theory is the eminent one due to its simplicity and ease of applicability in major area of mathematics. Following Banach Contraction principle, Boyd and Wong[5] investigated the fixed point results in nonlinear contraction mappings. The study of fixed point results in partially ordered sets was initiated by Ran and Reurings[22]. Their results are hybrid of two classical theorems. Banach fixed point theorem and Knaster-Tarski fixed point theorem. Neito and Rodriguez-Lopez[15,16] extended the main results of Ram and Reurings showing that monotonicity and continuity are not necessary for uniqueness of fixed point. Eventually, many authors extended and generalized this fixed point theorem from different points of view.

There have been many exciting developments in the field of existence and uniqueness of fixed points in various directions[1, 4, 6, 11, 12, 13, 14, 18, 26, 27].

Recently, Parvaneh[17] proved some common fixed point theorems for weakly compatible pair of mapping in the set up of complete metric space. Zhang and Song[30] proved some common fixed point theorems for two single valued generalized ϕ-weak contraction mappings. In this paper, we prove some common fixed point theorems for generalized ϕ-weak contraction mappings using the notion of weak compatibility. Also we prove a common fixed point theorem for A_ϕ contraction mapping in the setting of complete metric space.

2. PRELIMINARIES

Definition 2.1 [3] "A self mapping $T : X \rightarrow X$ on a metric space (X, d) is said to be a ϕ-weak contraction if there exists a map $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))."$$

In 1997, Alber and Guerre-Delabriere[3] defined the concept of ϕ-weak contraction. Also, Rhodes[23] proved the following fixed point theorem for ϕ-weak contraction mapping, which is one of the generalization of Banach contraction principle.

Theorem 2.2 [23] Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping on X such that for all $x,y \in X$, we have

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non decreasing function with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

These concept of generalized ϕ-weak contraction was introduced by Zhang and Song in 2009.

Definition 2.3 [30] Let (X, d) be a metric space. Two self mappings $S, T : X \rightarrow X$ are said to be generalized ϕ-weak contractions if there exists a map $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$

$$d(Tx, Sy) \leq N(x, y) - \phi(N(x, y)),$$

where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(y, Tx))\}.$$

Zhang and Song proved the following theorem for two single valued generalized ϕ-weak contraction mappings.

Theorem 2.4 [30] Let (X, d) be a metric space and S, T : $X \to X$ be two mappings such that for all $x,y \in X$

$$d(Tx, Sy) \leq N(x, y) - \phi(N(x, y))$$

where $\phi : [0, \infty) \to [0, \infty)$ is a lower semi continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T and S have a unique common fixed point.

Definition 2.5 [24] Consider he class of functions $\phi = \{ \phi \mid \phi : R_+ \to R_+\}$, which satisfies the following assertions :

(i) $t_1 \leq t_2$ implies $\phi(t_1) \leq \phi(t_2)$,

(ii) $(\phi^n(t))_{n \in N}$ converges to 0 for all $t > 0$,

(iii) $\sum \phi^n(t)$ converges for all $t > 0$.

If conditions (i – ii) hold then ϕ is called a comparison function, and if the comparison function satisfies (iii), then ϕ is called a strong comparison function.

Remark 2.6 [24] Any strong comparison function is a comparison function.

Remark 2.7 [24] If $\phi : R_+ \to R_+$ is a comparison function, then $\phi(t) < t$, for all $t > 0$, $\phi(0) = 0$ and ϕ is right continuous at 0.

Definition 2.8 [2] Suppose R_+ is the set of all non negative real numbers and A be the collection of all functions $\alpha : R_+^3 \to R_+$ which satisfies the following conditions:

(i) α is continuous on R_+^3 (with respect to the Euclidean metric on R_+^3),

(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all a, b

Definition 2.9 [2] Let (X, d) be a metric space and T a self map on X. Then T is said to be a A-contraction if

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in A$.

Definition 2.10 [9] Let (X, d) be a metric space and T, S are two self maps on X. T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx$ implies $TSx = STx$.

Definition 2.11. Let R_+ be the set of all non-negative real numbers and A_ϕ be the collection of all functions $\alpha : R_+^3 \to R_+$ which satisfies the following conditions :

(i) α is continuous on R_+^3 (with respect to the Euclidean metric on R_+^3).

(ii) For all $u, v \in R_+$. $u \leq \alpha(u, v, v)$ or $u \leq \alpha(v, u, v)$ or $u \leq \alpha(v, v, u)$, then $u \leq \phi(v)$, where ϕ is a strong comparison function.

When $\phi(t) = kt$ as $k\in(0, 1)$ for all $t > 0$, then we have $\alpha\in A$.

3. MAIN RESULT

Definition 3.1 – Let (X, d) be a metric space and A, B, S and T be self mappings on X. we define

$$M(x, y) = \max\{d(Sx, Ty), d(Sx\ Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\}$$

Theorem 3.2 :- Let (X, d) be a complete metric space. Let A, B, S and T be self maps on X such that for all $x, y\in X$ with $x \neq y$.

$$d(Ax, By) \leq M(x, y) - \phi(M(x, y)), \tag{3.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is lower semi continuous with $0 < \phi(t) < t$ for $t \in(0, \infty)$ and $\phi(0) = 0$. These mappings satisfying the following assertions:

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, and T(X) or S(X) is a closed subset of X.

(ii) The pair (A, S) and (B, T) are weakly compatible.

Then A, B, S, and T have a unique common fixed point in X.

Proof :- Let $x_0\in X$ be an arbitrary point

Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

then there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$ and for $x_1\in X$ such that $Ax_0 = Tx_1$ and for $x_1\in X$, there exists a point $x_2\in X$ such that $Bx_1 = Sx_2$.

Inductively, we can construct a sequence.

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$$
$$y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \qquad \text{for } n = 0, 1, 2,\ldots$$

We assume, for all $n \in N \cup \{0\}$,

$$y_{2n} \neq y_{2n+1} \tag{3.2}$$

At first, we shall show that $d(y_{2n}, y_{2n+1}) \to 0$ as $n\to\infty$ for all $n\in N\ U\{0\}$, where N is a set of natural numbers.

For this, suppose that $x = x_{2n}$, $y = x_{2n+1}$ in (3.1), we have

$$d(Ax_{2n}, Bx_{2n+1}) = d(y_{2n+1}, y_{2n+2})$$
$$\leq M(x_{2n}, x_{2n+1}) - \phi(M(x_{2n}, x_{2n+1})) \tag{3.3}$$

where

$$M(x_{2n}, x_{2n+1}) = \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1})$$
$$d(y_{2n+1}, y_{2n+2}), \frac{1}{2}(d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1}))\}$$

Then, by the triangular inequality, we have

$$M(x_{2n}, x_{2n+1}) \leq \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+2}), \frac{1}{2}(d(y_{2n}, y_{2n+1}) +$$
$$d(y_{2n+1}, y_{2n+2}))\}$$

If $d(y_{2n}, y_{2n+1}) < d(y_{2n+1}, y_{2n+2})$
then we get

$$M(x_{2n}, x_{2n+1}) \leq d(y_{2n+1}, y_{2n+2}) \quad (3.4)$$

and (3.3) implies the following

$$d(y_{2n+1}, y_{2n+2}) \leq M(x_{2n}, x_{2n+1}) - \phi(M(x_{2n}, x_{2n+1}))$$

Using the monotonically increasing property, we have

$$d(y_{2n+1}, y_{2n+2}) \leq M(x_{2n}, x_{2n+1}) \quad (3.5)$$

From (3.4) and (3.5) we have

$$M(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2}) \quad (3.6)$$

Since $0 < |d(y_{2n+1}, y_{2n+2}) - d(y_{2n}, y_{2n+1})| \leq d(y_{2n}, y_{2n+2})$ (3.7)

We have $M(x_{2n}, x_{2n+1}) > 0$, then from (3.3), (3.6) and the property of ϕ functions, we have

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+1}, y_{2n+2}) - \phi(d(y_{2n+1}, y_{2n+2})) < d(y_{2n+1}, y_{2n+2})$$

Which is a contradiction. Thus we have

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}) \quad (3.8)$$

So, we obtain the following

$$M(x_{2n}, x_{2n+1}) = d(y_{2n}, y_{2n+1}) \quad (3.9)$$

Now putting (3.9) in (3.3), we have

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1}) - \phi(d(y_{2n}, y_{2n+1})) \quad (3.10)$$

Therefore $d(y_{2n}, y_{2n+1})$ is a monotonically decreasing sequence of non negative real numbers, then there exists a number $r > 0$ such that

$$\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = r > 0 \quad (3.11)$$

Taking $n\to\infty$ in (3.10) and using (3.11), we get

$$\lim_{n\to\infty} d(y_{2n+1}, y_{2n+2}) \leq \lim_{n\to\infty} d(y_{2n}, y_{2n+1}) - \lim_{n\to\infty} \phi(d(y_{2n}, y_{2n+1}))$$

This implies that

$$r \leq r - \phi(r) \Rightarrow \phi(r) \leq 0$$

We observe that the last term on the right hand side of the above inequality is non-zero. We get a contradiction with ϕ function. Therefore, we have

$$\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$$

Putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (3.1) and arguing as above, we obtain

$$\lim_{n\to\infty} d(y_{2n+1}, y_{2n+2}) = 0$$

Therefore, for all $n \in N \cup \{0\}$, we have

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \quad (3.12)$$

Next, we prove that $\{y_n\}$ is a Cauchy sequence. For this, it is enough to show that the subsequence $\{y_{2n}\}$ is a Cauchy sequence. To the contrary, suppose that $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\in > 0$ and the sequence of natural numbers $\{2n(k)\}$ and $\{2m(k)\}$ such that $2n(k) > 2m(k) > 2k$ for $k \in N$ and

$$d(y_{2m(k)}, y_{2n(k)}) \geq \in \tag{3.13}$$

Corresponding to 2m(k). We can choose 2n(k) to be the smallest such that (3.13) is satisfied. Then we have

$$d(y_{2m(k)}, y_{2n(k)-1}) < \in \tag{3.14}$$

Putting $x = x_{2m(k)-1}$ and $y = x_{2n(k)-1}$ in (3.1), where for all $k \in N$,

$$d(y_{2m(k)}, y_{2n(k)}) \leq M(x_{2m(k)-1}, x_{2n(k)-1}) - \phi\,(M(x_{2m(k)-1}, x_{2n(k)-1}) \tag{3.15}$$

where

$$M(x_{2m(k)-1}, x_{2n(k)-1}) = \max\{d(y_{2m(k)-1}, y_{2n(k)-1}), d(y_{2m(k)-1}, y_{2m(k)}), d(y_{2n(k)-1}, y_{2n(k)}), \frac{1}{2}(d(y_{2m(k)-1}, y_{2n(k)}) + d(y_{2n(k)-1}, y_{2m(k)}))\}$$

Using the triangle inequality, we have

$$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})$$

Taking the limit $k \to \infty$, we get

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \in \tag{3.16}$$

Again for all k, we have

$$d(y_{2m(k)-1}, y_{2n(k)-1}) \leq d(y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)}) + d(y_{2n(k)}, y_{2n(k)}),$$

On letting limit $k\to\infty$ and using (3.12) and (3.16), we get

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) = \in \tag{3.17}$$

Again for all positive integer k, we have

$$d(y_{2m(k)-1}, y_{2n(k)}) \leq d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2n(k)})$$

Letting limit $k\to\infty$ and using (3.12) and (3.17), we get

$$\lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) = \in \tag{3.18}$$

Again for all positive integers k, we have

$$d(y_{2n(k)-1}, y_{2m(k)}) \leq d(y_{2n(k)-1}, y_{2n(k)}) + d(y_{2n(k)}, y_{2m(k)})$$

Letting limit $k\to\infty$ and using (3.12) and (3.18), we get

$$\lim_{k\to\infty} d(y_{2n(k)-1}, y_{2m(k)}) = \in \tag{3.19}$$

From (3.15) – (3.19), we get

$$\lim_{k\to\infty} M(x_{2m(k)-1}, x_{2n(k)-1}) = \in$$

Letting $k\to\infty$ in (3.15), we get

$$\in \leq \in - \phi(\in)$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that the last term on the right hand side of the above inequality is non-zero. Thus we arrive at a contradiction.

Hence $\{y_n\}$ is a Cauchy sequence.

Therefore, the Cauchy sequence $\{y_n\}$ is a convergent sequence, and it converges to a point z (say) in X.

Consequently, the sub-sequences also converges to z in X.

$$Ax_{2n} \to z,\ Tx_{2n+1} \to z,\ Bx_{2n+1} \to z \text{ and } Sx_{2n} \to z.$$

Now we shall prove that z is a common fixed point of A, B, S and T.

Since $B(X) \subset S(X)$, there exists $v \in X$ such that $z = Sv$. Let $d(z, Av) \neq 0$. Putting $x = v$ and $y = x_{2n+1}$ in (3.1), we get

$$d(Av, Bx_{2n+1}) \leq M(v, x_{2n+1}) - \phi(M(v, x_{2n+1})) \tag{3.20}$$

where

$$M(v, x_{2n+1}) = \max\{d(Sv, Tx_{2n+1}), d(Sv, Av),$$
$$d(Tx_{2n+1}, Bx_{2n+1}), \frac{1}{2}(d(Sv, Bx_{2n+1}) + d(Tx_{2n+1}, Av))\}$$

Taking $n \to \infty$ and using $z = Sv$, we have

$$M(v, z) = \max\{d(Sv, z), d(Sv, Av), d(z, z), \frac{1}{2}(d(Sv, z) + d(z, Av))\}$$
$$= \frac{1}{2}(d(z, Av))$$

Also, we have

$$d(Av, z) \leq \frac{1}{2}(d(z, Av)) - \phi\left(\frac{1}{2}(d(z, Av))\right)$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, therefore, we obtain

$$d(Av, z) < \frac{1}{2}(d(z, Av))$$

Hence we arrive at a contradiction.

Therefore

$$d(z, Av) = 0 \quad \Rightarrow \quad Av = z \quad \Rightarrow \quad Av = z = Sv$$

Since (A, S) is a weakly compatible pair of maps, so it commutes at their coincidence point v, i.e., $ASv = SAv \quad \Rightarrow Az = Sz$

Now, we shall prove that $Az = Sz = z$.

For this, putting $x = z$ and $y = x_{2n+1}$ in (3.1), we get

$$d(Az, Bx_{2n+1}) \leq M(z, x_{2n+1}) - \phi(M(z, x_{2n+1})) \tag{3.21}$$

where

$$M(z, x_{2n+1}) = \max\{d(Sz, Tx_{2n+1}), d(Sz, Az)$$
$$d(Tx_{2n+1}, Bx_{2n+1}), \frac{1}{2}(d(Sz, Bx_{2n+1}) + d(Tx_{2n+1}, Az))\}$$

Taking $n \to \infty$ and using $Az = Sz$, we get

$$M(z, z) = d(Sz, z)$$

Now (3.21) implies that

$$d(Sz, z) \leq d(Sz, z) - \phi(d(Sz, z))$$

Using discontinuity of ϕ at $t = 0$ and $\phi(t) > 0$ for $t > 0$, we observe that

$$d(Sz, z) < d(Sz, z)$$

Which is contradiction. Therefore $d(Sz, z) = 0 \Rightarrow Sz = z \Rightarrow Sz = Az = z$. Similarly, we can show that $Tz = Bz = z$

Hence $Sz = Az = Tz = Bz = z$.

Now, we shall show that z is the unique common fixed point of A, B, S and T.

Let z_1 be also a fixed point of A, B, S and T.

Putting $x = z$ and $y = z_1$ in (3.1), we have

$$d(z, z_1) \leq d(z, z_1) - \phi(d(z, z_1)) \tag{3.22}$$

a contradiction. Hence $d(z, z_1) = 0 \Rightarrow z = z_1$.

Hence A, B, S and T have a unique common fixed point in X.

When we take $S = T = I$ identity map, we get the following theorem.

Theorem 3.3 Let (X, d) be a complete metric space let $A, B : X \to X$ be two self mapping which satisfy the following inequality.

$$d(Ax, By) \leq M(x, y) - \phi(M(x, y)) \tag{3.23}$$

where $x, y \in X, x \neq y$

$$M(x, y) = \max\{d(x, y), d(x, Ax), d(y, By), \frac{1}{2}(d(x, By) + d(y, Ax))\}$$

(i) $\phi : [0, \infty) \to [0, \infty)$ is such that $\phi(t) > 0$ which is lower semi-continuous for all $t > 0$, and ϕ is discontinuous at $t = 0$ with $\phi(0) = 0$.

Then there exists a unique fixed point of A, B and X.

Theorem 3.4 : Let (X, d) be a complete metric space. Let A, B, S and T be self maps on X. If there exists $\alpha \in A_\phi$ such that for all $x, y \in X$ with $x \neq y$

$$d(Ax, By) \leq \alpha(d(Sx, Ty), d(Sx, Ax), d(Ty, By)) \tag{3.24}$$

satisfying the following assertions:

(i) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, and T(X) or S(X) is closed subset of X.

(ii) The pair (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.

Proof :- Let $x_0 \in X$ be an arbitrary point. Define a sequence

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$$
$$y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \qquad \text{for } n = 0, 1, 2,\ldots$$

We assume that, for all $n \in N \cup \{0\}$

$$y_{2n} \neq y_{2n+1}$$

At first, we shall show that $d(y_{2n}, y_{2n+1}) \to 0$ as $n \to \infty$ for all $n \in N \cup \{0\}$, where N is a set of natural numbers.

For this, suppose that $x = x_{2n}$, $y = x_{2n+1}$ in (3.24), we have

$$d(Ax_{2n}, Bx_{2n+1}) = d(y_{2n+1}, y_{2n+2})$$
$$\leq \alpha(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}))$$

By the definition of α,

$$d(y_{2n+1}, y_{2n+2}) \leq \phi(d(y_{2n}, y_{2n+1}))$$

Similarly

$$d(y_{2n+2}, y_{2n+3}) \leq \alpha(d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3}))$$

By the definition of α,

$$d(y_{2n+2}, y_{2n+3}) \leq \phi(d(y_{2n+1}, y_{2n+2})$$

Combining this way, we get

$$\begin{aligned} d(y_{2n+2}, y_{2n+3}) &\leq \phi(d(y_{2n+1}, y_{2n+2})) \\ &< \phi(\phi(d(y_{2n}, y_{2n+1}))) \\ &= \phi^2(d(y_{2n}, y_{2n+1})) \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ &\leq \phi^{2n+2}(d(y_0, y_1)) \end{aligned}$$

Thus $\quad d(y_n, y_{n+1}) \leq \phi^n(d(y_0, y_1))$

For all $n \in N$

Since $d(y_0, y_1) \geq 0$. So from definition 2.5(ii), we have $\lim_{n\to\infty} \phi^n(d(y_0, y_1)) = 0$.

Now, for a given $\in > 0$, there is a positive integer n_0 such that $n \geq n_0$,

$$\phi^n(d(y_0, y_1)) < \in - \phi(\in)$$

Hence

$$d(y_n, y_{n+1}) < \in - \phi(\in) \tag{3.25}$$

Now, for any $m, n \in N$ with $m > n \geq n_0$, we claim that

$$d(y_n, y_m) < \in \tag{3.26}$$

We prove this by induction on m. The inequality holds for $m = n+1$ by using equation (3.25). Assume that inequality (3.26) hold for $m = k$. i.e., $d(y_n, y_m) < \in$.

Now if $m = k + 1$, we have

$$\begin{aligned} d(y_n, y_{n+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{k+1}) \\ &< \in - \phi(\in) + \phi(d(y_n, y_k)) \\ &< \in - \phi(\in) + \phi(\in) \\ &= \in \end{aligned}$$

By induction on m, we conclude that the inequality (3.26) holds for $m > n \geq n_0$.

Thus $\{y_n\}$ is a Cauchy sequence. Since (X, d) is complete and $\{y_n\}$ is Cauchy in X, so there is a $z \in X$ such that $\lim_{n\to\infty} y_n = z$ and $z \in X$. By assumption S(X) is closed, so there exists $u \in X$ such that $S(u) = z$.

Now $\quad d(Au, y_{n+1}) \leq \alpha(d(Su, Tx_{2n+1}), d(Su, Au), d(Tx_{2n+1}, Bx_{2n+1}))$

$$= \alpha(d(z, y_{2n+2}), d(z, Au), d(y_{2n+2}, y_{2n+1}))$$

If $n \to \infty$

$$d(Au, z) \leq \alpha(0, d(z, Au), 0)$$

Hence $d(Au, z) \leq \phi(0) = 0$. Thus $Au = Bz$.

Similarly, we can show that $Bu = z$.

Therefore $Au = Bu = Su = Tu = z$, since the pair (A, S) and (B, T) are weakly compatible, we have

$$Az = Bz = Sz = Tz.$$

Now, we have

$$\begin{aligned} d(Az, y_{2n+1}) &= d(Az, Bx_{2n+1}) \\ &\le \alpha(d(Sz, Tx_{2n+1}), d(Sz, Az), d(Tx_{2n+1}, Bx_{2n+1}) \\ &= \alpha\, (d(Az, y_{2n+1}), d(Az, Az), d(y_{2n+1}, y_{2n+2}) \end{aligned}$$

Taking limit $n \to \infty$

$$d(Az, z) \le \alpha(d(Az, z), 0, 0))$$

Thus $d(Az, z) \le \phi(0) = 0$, which gives $Az = z$.
Then we conclude that $Az = Bz = Sz = Tz = z$.
So, z is a common fixed point of A, B, S and T.
If there exist another fixed point $v \in X$ such that $v = Av = Bv = Sv = Tv$, then we get

$$\begin{aligned} d(z, v) &= d(Az, Bv) \\ &\le \alpha(d(z, v), d(Sz, Az), d(Tv, Bv)) \\ &= \alpha(d(z, v), 0, 0). \end{aligned}$$

Thus $d(z, v) \le \phi(0)$, i.e., $z = v$. Hence A, B, S and T have a unique fixed point.

CONCLUSION

In this paper, using the notion of weak compatibility, we have extended some common fixed point theorems for generalized ϕ-weak contraction and A_ϕ contractions defined on a complete metric space. The result discussed in this paper are mainly concerned with the existence and uniqueness of common fixed point. Study of coincidence points and coupled coincidence points for these maps would also be interesting topic for future study.

REFERENCES

1. R.P. Agarwal, D. O'Regan, Fixed point theory for generalized contractions on spaces with two metrices, *J. Math. Anal. Appl*., 248, no. 2, 402-414 (2000).
2. M.Akram, A.A. Zafar, A.A. Siddiqui, A general class of contractions : A-contractions, *Novi Sad J. Math*.,38, 25-33 (2008).
3. Y.I. Alber, S. Guerre-Delabriere, Principles of weakly contractive maps in Hilbert spaces, *New results in Operator Theory, Advances and Appl*., 98, 7-22 (1997).
4. N.A. Assad, W.A. Kirk, Fixed point theoresm for set-valued mappings of contractive type, *Pac. J. Math*.,43(3), 553-561 (1972).
5. D.W. Bovd., J.S. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc*.,20, 458-464 (1969).
6. S.K. Chatterjea, Fixed point theorems, *C.R. Acad, Bulgare Sci*., 25, no. 6, 727-730 (1972).
7. D. Doric, Common fixed point for generalized (ϕ, ψ)-weak contractions, *Appl. Math. Lett*., 22(12), 1806-1900 (2009).
8. J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, *Nonlinear Anal*., 74(3), 768-774 (2011).
9. G. Jungck. B.E. Rhoades, Fixed point for set valued functions without continuity, *Indian J. Pure Appl. Math*., 29, no. 3, 227-238 (1998).

10. Z. Kadelburg, M. Pavlovic, S. Radenovic, Common fixed point theorems for ordered contractions and quasi contractions in ordered cone metric spaces, *Comput. Math. Appl*., 59(9), 3148-3159 (2010).
11. R. Kannan, Some results on fixed points, *Bull. Calc. Math. Soci*., 60, no. 1, 71-77 (1968).
12. V. Lakshmikantham and L. Ciric. Counted fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonlinear Anal. Theory Methods Appl*., 70, no. 12, 4341-4349 (2009).
13. S.B. Nadler, Multi-valued contraction mappings, *Pac. J. Math*., 30(2), 475-488 (1969).
14. J.J. Nieto, R.L. Pouso, R. Rodiriguez-Lopez, Fixed point theorems in ordered abstract spaces, *Proc. Amer. Math. Soci*., 135, 2505-2517 (2007).
15. J.J. Nieto R. Rodriguez-Lopez, Contraction mapping theorems in partially ordered sets and applications to ordeinary differential equations, *Order*, 22, no. 3, 223-239 (2005).
16. J.J. Nieto, R. Rodiriguez-Lopez, Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Act. Math. Sin., English Series*, 23, no. 12, 2205-2212 (2007).
17. V. Parvaneh, Some common fixed point theorems in complete metric spaces, *Int. J. Pure Appl. Math*., 76, no. 1, 1-8 (2012).
18. A. Petrusel, I.A. Rus, Fixed point theorems in ordered L-spaces, *Proc. Amer. Math. Soci*., 134, 411-418 (2006).
19. P. Popeseu, Fixed points for (ϕ, ψ)-weak contractions, *Appl. Math. Lett*., 24(1), 1-4 (2011).
20. S. Radenovic, Z. Kadelburg, Generalized weak contractions in partially ordered metric spaces, *Comput. Math. Appl*. 60, 1776-1783 (2010).
21. S Radenovic, Z. Kadelburg, D. Jandrlic, A. Jandrlic, Some results on weakly contractive maps, *Bull. Iranian Math. Soci*., 38, 625-645 (2012).
22. A.C. M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to metric equations, *Proc., Amer. Math. Soci*.,132, no. 5, 1435-1443 (2004).
23. B.E. Rhoades, Some theorem on weakly contractive mapos. *Nonlinear Anal*., 47, 2683-2693 (2001).
24. I.A. Rus, N. Petrusel, G. Pentrusel, Fixed point Theory, Cluj Univ. Press., (2008).
25. N. Shahzad, A.F.R.L. Hierro, F. Khojasteh. Some fixed point theorems under (A, S)-contractively conditions, *RACSAM*, 111, no. 2, 307-324 (2017).
26. W. Sintumavarat, Petrusel, P. Kumam, Common coupled fixed point theorems for ω*-compatible mappings without mixed monotonic property, Rendiconti del Circolo Matematico di Palermo. 61, no. 3, 361-383 (2012).
27. W. Sintumavarat. S. Radenovic Z. Glubovic, P. Kumam, Coupled fixed point theorems for F-internal sci, *Appl. Math. Int. Sci.,* 7, no. 1, 247-255 (2013).
28. T. Van An. N. Van Dung. Z. Kadelburg, S. Radenvoic, Various generalizations of metric spaces and fixed point theorems, *RACSAM*, 109, no. 1, 175-198 (2015).
29. Z. Wu. C. Zhu, C. Yuan, Fixed point results for cyclic contractions in Menger PM-spaces and generalized menger PM-spaces, FACSAM doi : 10.1007/s13398-017-0393-8 (2017).
30. Q. Zhang, Y. Song, Fixed point theory for generalized ϕ-weak contractions, *Appl. Math. Lett*., 22, 75-78 (2009).