Journal of Computer and Mathematical Sciences, *Vol.8(9), 442-451 September 2017* *ISSN 0976-5727 (Print)*
(An International Research Journal), *www.compmath-journal.org* *ISSN 2319-8133 (Online)*

A Study on Equitable Irregular Edge-Weighting of Graphs

Meenakshy Rajan[1] and Abraham V. M.[2]

Department of Mathematics,
Christ University, Bengaluru, INDIA.
email: meenakshyrajan7@gmail.com,
frabraham@christuniversity.in

(Received on: August 24, Accepted: September 12, 2017)

ABSTRACT

Given a graph G = (V, E), a k-edge-weighting is a map Ø: E(G) → {1, 2, 3, ..., k}, where k is a positive integer. For a vertex v of G, let $S_Ø(v)$ denote the sum of edge weights appearing on the edges incident at v under the edge-weighting Ø, called the weighted degree of v. A nonproper k-edge-weighting of G is said to be equitable irregular if $|S_Ø(u) - S_Ø(v)| \leq 1$, for every pair of adjacent vertices u and v in G. A graph G is said to be equitable irregular if G admits an equitable irregular edge weighting. If G is equitable irregular then the equitable irregular strength of G is defined to be the smallest positive integer k such that G has a k-edge weighting and is denoted by S_e (G). In this paper, we carry out a study of this property for certain classes of graphs. We find some classes of equitable irregular graphs and find a sufficient condition for equitable irregularity. We find an algorithm for constructing equitable irregular trees with stars as base graphs and characterize the class of caterpillars for equitable irregularity. We determine some lobster graphs that are equitable irregular and characterize binary trees for this property.

Keywords: Equitable irregular graphs; Triangular ladder; Triangular book; Star graphs; Caterpillars; Lobsters; Binary trees.

1. INTRODUCTION

In this study, we deal only with simple, undirected, connected graphs. For some basic terminologies we have referred West[7], Chartrand and Zhang[2] and Hamid and Kumar[5]. Graph labelling is the process of assigning labels to the vertices or edges or both of a graph. The labels we use predominantly are integers. The origin of graph labelling dates back to 1967 when Rosa[6], studied this concept. Rosa identified the three types of labelling - α, β- and ρ -

labelling. In the following years many types of graph labelling were studied in more than 800 articles. Gallian[3] has made a detailed survey of graph labelling later.

In this study, we are dealing with edge labelling, the assignment of labels to the edges of a graph. We use the term edge weight instead of edge label. We deal with non-proper edge weighting in our study, that is adjacent edges need not get different weights. Irregularity strength is one of the parameters first studied in this area. This study was initiated by Chartrand *et al*.[1]. Let G be a graph and Ø be a *k*-edge weighting of G. We denote by $S_{\emptyset}(v)$ the term weighted degree of a vertex *v* of G, the sum of all edge weights appearing on the edges incident with *v*. One of the initial problems discussed in this area was to find the smallest integer *k* for which a *k*-edge weighting Ø of G such that for all vertices x and y of G, $S_{\emptyset}(x) \neq S_{\emptyset}(y)$. In other words, the weighted degrees of the vertices of G are all distinct. This graph parameter is called the irregularity strength of G and is denoted by S(G) and the k-edge weighting is called irregular edge-weighting. Motivated by the results on irregularity strength, Karonski *et al*.[4] proposed the study of non-proper edge-weightings where they require that only adjacent vertices have distinct weighted degrees and called such edge-weighting as chromatic irregular edge weighting. Thus, in an irregular edge-weighting of a graph G, the weighted degrees of the vertices of G admit. Whereas in chromatic irregular edge-weighting only adjacent vertices have different weighted degrees. Sahul Hamid and Ashok Kumar[5] initiated the study on equitable irregular graphs. They have identified some graphs that are equitable irregular and presented some properties of such graphs. This paper is a continuation of this work and covers equitable irregularity of certain graphs and studies in detail the same on classes of trees.

2. EQUITABLE IRREGULAR GRAPHS

Definition 2.1. Given a graph G = (V, E), a *k*-edge-weighting is a map Ø: E(G) → {1, 2, 3, ..., *k*}, where *k* is a positive integer. For a vertex *v* of G, let $S_{\emptyset}(v)$ denote the sum of edge weights appearing on the edges incident at *v* under the edge-weighting Ø, called the **weighted degree** of *v* [4]. A nonproper *k*-edge-weighting of G is said to be **equitable irregular** if $|S_{\emptyset}(u) - S_{\emptyset}(v)| \leq 1$, for every pair of adjacent vertices *u* and *v* in G. A graph G is said to be equitable irregular if G admits an equitable irregular edge weighting. If G is equitable irregular then the **equitable irregular strength** of G is defined to be the smallest positive integer *k* such that G has a *k*-edge weighting and is denoted by S_e (G)[5].

Example 2.1. We list some examples for equitable and non-equitable irregular graphs.

1. All regular graphs are equitable irregular with equitable irregular strength 1.
2. All paths P_n are equitable irregular with $S_e(P_n) = 1$
3. A grid P_m, P_n is equitable irregular with equitable irregular strength 1.
4. The graph G given below is not equitable irregular.

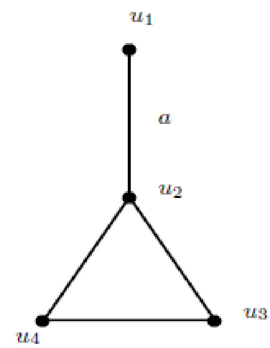


Figure 1: G

3. SOME CLASSES OF EQUITABLE IRREGULAR GRAPHS

Definition 3.1. For a two graphs G and H, by attaching H to a vertex v of G, we mean pasting any arbitrary vertex *u* of H to *v*.

Example 3.1. Consider the graphs G and H given in the following figure.

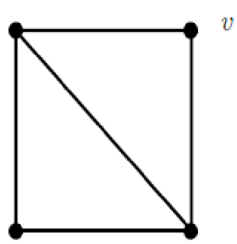


Figure 2: G

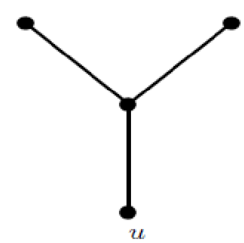
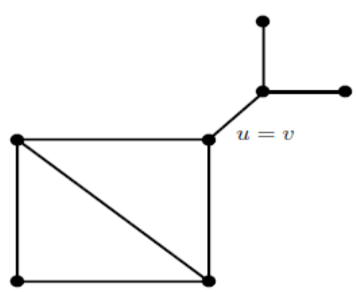


Figure 3: H

The graph obtained by attaching H at the vertex v of G is given by



Definition 3.2. For a connected graph G and a positive integer n ≥3 the graph G < C_n > is obtained from G by attaching a copy of the cycle C_n at each vertex of G.

Example 3.2. For the graph G given in Figure 2, G <C_4 > is given by

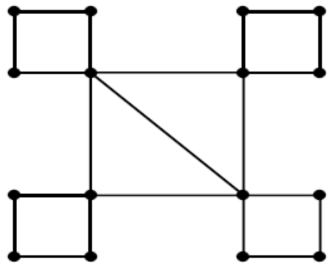


Figure 4: $G < C_4 >$

Sahul Hamid and Ashok Kumar[5] have already proved that for any graph G, the graph obtained by attaching C_3 to all vertices of G generates an equitable irregular graph. We generalize this result for all odd cycles C_n.

Theorem 3.1. For any graph G, the graph G <C_n > is equitable irregular if *n* is odd.

Proof. Let V(G) = $\{v_1, v_2, \dots, v_n\}$. Let the cycle C_i attached at v_i be $C_i := (v_i = x_{i0}, x_{i1}, \dots, x_{i\,n-1})$. We wish to prove that G <C_n > is equitable irregular. Without loss of generality assume x_{10} is pasted on to v_i for all i = 1, 2, …, k. We define an edge weighting Ø on G <C_n > as follows. Let Ø(e) = 1 for all e ϵ E(G). For each i = 1, 2, …, k,

Ø($x_{i(2j-1)}\, x_{i(2j)}$) = floor ($\frac{\Delta(G) - deg_G(v_i) + 2}{2}$) = X_i , say for j = 1, 2, …, $\frac{n-1}{2}$.

Ø($x_{i(2j)}\, x_{i(2j+1)}$) = ceil($\frac{\Delta(G) + deg_G(v_i) + 1}{2}$) = Y_i , say for j = 1, 2, …, $\frac{n-1}{2}$.

For i = 1, 2, …, k, when we expand we can see that $S_Ø(v_i)$ = deg(v_i) + 2 X_i, which is either Δ(G) + 1 or Δ(G) + 2.

For j = 1, 2, …, $\frac{n-1}{2}$, $S_Ø(x_{i(2j)})$ = $S_Ø(x_{i(2j+1)})$= $X_i + Y_i$, which also is either Δ(G) + 1 or Δ(G) + 2.

Thus, the weighted degrees of any two adjacent vertices differ by at most 1, so G < Cn > is equitable irregular.

Example 3.3. The following figure shows a graph G and the corresponding graph G < C_5 > which is equitable irregular.

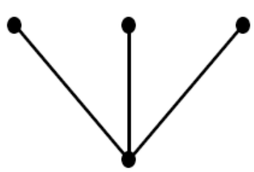


Figure 5: G

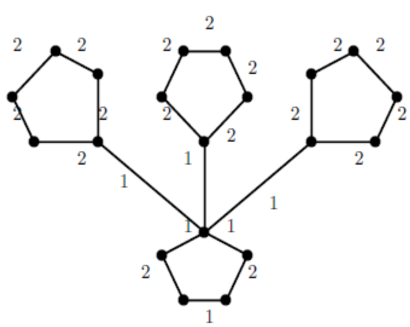


Figure 6: $G < C_5 >$

Definition 3.3. For a connected graph G, the graph G < C > is obtained from G by attaching a copy of a cycle C at each vertex of G.

Remark: The difference between G < C_n > and G < C > is that in G < C_n > at each vertex we have the same cycle C_n but in G < C > at each vertex the length of the cycle may differ. In short G < C_n > is a special case of G < C > in which the cycles attached at all the vertices have equal length.

Theorem 3.2. For any graph G, the graph G < C > is equitable irregular if C is an odd cycle.
Proof. The edge weighting _ mentioned in Theorem 3.1 yields an equitable irregular edge weighting For G < C > too.

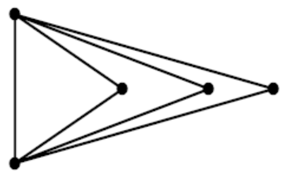


Figure 7: $K_{1,1,3}$

Definition 3.4. A triangular book graph $K_{1,1,n}$ [3] is a graph consisting of n triangles sharing a common edge. Each triangle is called a page of the book.

Example 3.4. The following figure shows a triangular book $K_{1,1,3}$

Theorem 3.3. The triangular book $K_{1,1,n}$ is equitable irregular if and only if n≤2.
Proof. Let G = $K_{1,1,n}$. Suppose n ≤2. If Ø is an edge weighting of G which assigns 1 to all edges of the graph, then clearly G is equitable irregular since G is a semi regular graph for n = 1, 2. Now suppose n ≥ 3. If *u* and *v* are the endpoints of the common edge of the pages of the book, then, $S_Ø$ (u) = $S_Ø$ (v) ≥ 4 and $S_Ø$ (x_i) ≥2 for all the other vertices x_i, *i* = 1, 2, …, n. Thus, the difference in the weighted degrees differ by at least 2. So, G is not equitable irregular.

Definition 3.5. A triangular ladder TL_n, n ≥ 2 [10] is obtained from a ladder L_n= P_n□ P_2 by adding edges between u_i and v_{i+1} , 1≤ *i* ≤ *n-1*.

Example 3.5. The following figure gives the triangular ladder TL_5.

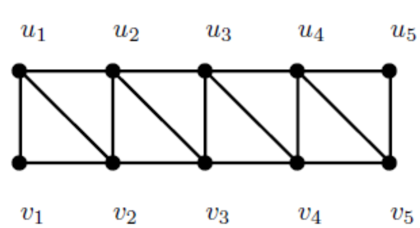


Figure 8: TL_5

Theorem 3.4. The triangular ladder TL_n is equitable irregular for all n ≥ 2.
Proof. Consider an edge weighting Ø on TL_n as follows.
Ø (u_1v_1) = Ø (u_nv_n) = 2
Ø (u_iu_{i+1}) = Ø(v_iv_{i+1}) = 1 for all 2 ≤ i ≤n -1
Ø (u1vi+1) = 1 for all 1 ≤i ≤n - 1
Then, $S_Ø$ (u_1) = Ø(u_1u_2) + Ø(u_1v_1) + Ø(u_1v_2) = 1+2+1 = 4,
$S_Ø(v_n)$ = Ø$(v_{n-1}v_n)$ + Ø (u_nv_n) + Ø$(u_{n-1}v_n)$ = 1 + 2 + 1 = 4,
$S_Ø(v_1)$ = Ø(u_1v_1) + Ø(v_1v_2) = 2 + 1 = 3 and
$S_Ø(u_n)$ = Ø$(u_{n-1}u_n)$ + Ø(u_nv_n) = 1 + 2 = 3.
For 2 ≤ i ≤n - 1, $S_Ø$ (u_i)=Ø$(u_{i-1}u_i)$ + Ø(u_iu_{i+1}) + Ø(u_iv_i)+ Ø(u_iv_{i+1})+ = 1 + 1 + 1 + 1 = 4
and $S_Ø$ (v_i) = Ø$(v_{i-1}v_i)$ + Ø(v_iv_{i+1}) + Ø(u_iv_i)+ Ø(u_iv_{i+1})+ 1 + 1 + 1 + 1 = 4
Thus, TL_n is equitable irregular.
The following figure shows an equitable irregular edge weighting of TL_5 as illustrated in the theorem.

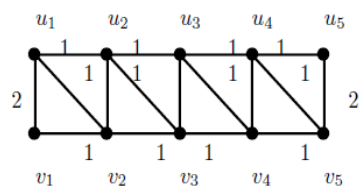


Figure 9: An equitable irregular weighting of TL_5

Now we present a property of equitable irregular graphs.

Proposition 3.1. In a graph G if the difference in degree of any pair of adjacent vertices is at most 1, G is equitable irregular.
Proof. Let G be a graph. Define an edge weighting Ø on G by Ø (e) = 1 for all e ϵ E(G).
So, $S_Ø$ (u)= deg(u) for all u ϵ V (G).
If *u*, *v* is a pair of adjacent vertices in G,|$S_Ø$ (u) - $S_Ø$ (v)|= |deg(u) - deg(v)|≤ 1.
Thus, G is equitable irregular.

Remark: The property mentioned in the above proposition gives a sufficient condition for equitable irregularity in graphs. Note that this condition is not necessary.

Example 3.6. Consider the graph G with the given edge weighting

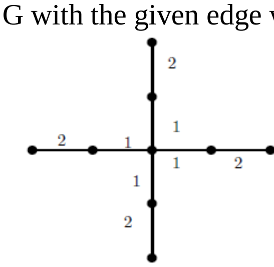


Figure 10: G

Here the degrees of all adjacent vertices of G does not differ by at most 1, still G is equitable irregular.

In the rest of the sections in this paper we deal only with trees and their equitable irregulari4.ty.

We consider some families of trees one by one and try to obtain the characterization for equitable irregularity in those families.

4. STAR GRAPHS AND EQUITABLE IRREGULARITY

Sahul Hamid and Ashok Kumar[5] have proved that if a graph G is equitable irregular, then every support vertex of G has degree 2, where a support vertex is a vertex adjacent to the pendant vertex. Based on this property we propose the following:

Proposition 4.1. The star graph $K_{1,n}$; n ≥3 is not equitable irregular.

Proof. In the Star graph $K_{1,n}$; n ≥3 the support vertex has degree greater than or equal to 3. Hence by the property mentioned above, $K_{1,n}$ is not equitable irregular for n ≥3.

Definition 4.1. A sub division[7] of an edge $e = uv$ of a graph G is obtained by replacing the edge *uv* with the path *u*, *w*, *v*.

Proposition 4.2. A sub division[7] of all edges of a star graph $K_{1,n}$ generates an equitable irregular graph.

Proof. Consider the graph $K_{1,n}$. Subdivide each of the n edges of $K_{1,n}$to obtain the graph G. Name the 2-degree vertices of G $u_1, u_2, \dots, u_n$ and the pendant vertices of G $v_1, v_2, \dots, v_n$ and let the root vertex be *w*. Let Ø(wu_i)= 1, i= 1, 2, …, n and Ø($u_i v_i$) = n - 2, i= 1, 2, … , n. Then clearly $|S_Ø(w) - S_Ø(u_i)|$= 1, i= 1, 2, …, n and $|S_Ø(v_i) - S_Ø(u_i)|$ = 1, i= 1, 2, …, n. Thus, the subdivisions of star graphs are equitable irregular.

Remark. The equitable irregularity strength of the above graph is S_e(G) = n-2.

We have seen that star graphs $K_{1,n}$; n ≥3 are not equitable irregular. We also saw by subdividing their edges we can construct equitable irregular graphs. There are more ways to construct trees that are equitable irregular from star graphs. Now we formulate an algorithm for constructing more equitable irregular trees from star graphs.

ALGORITHM FOR CONSTRUCTING EQUITABLE IRREGULAR TREES FROM STAR

Graphs

Step 1: Consider a star graph T_0 = $K_{1,n}$ n ≥ 2.

Step 2: For every *i*, construct $T_i$from T_{i-1} by attaching $K_{1,n-i}$ to each of the pendant vertices of T_{i-1}.

Step 3: Repeat Step 2 until *i* = n - 1 to obtain T_{n-1}

Proposition 4.3. The tree T_{n-1} in the above algorithm is equitable irregular.

Proof. Clearly from construction T_{n-1} is a tree. Define an edge weighting Ø on G byØ(e) = 1 for all e ϵ T_{n-1} This weighting makes T_{n-1} equitable irregular.
We illustrate below two trees T2 and T3, constructed from $K_{1,3}$ and $K_{1,4}$respectively using the algorithm mentioned above.

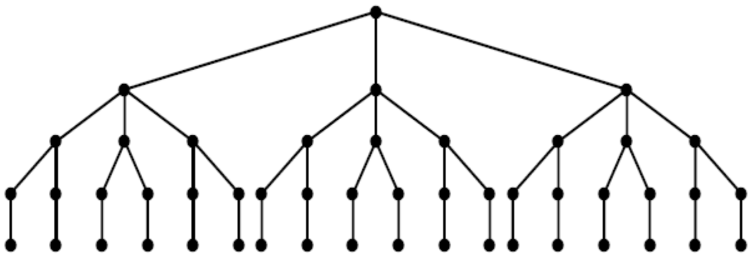


Figure 11: T_2

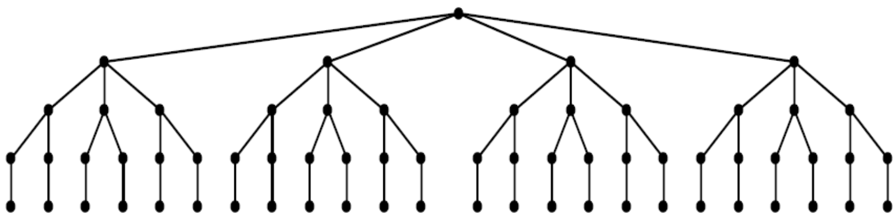


Figure 12: T_3

Remark

1. The equitable irregular strength of the tree T_{n-1} formed from the algorithm is 1.
2. In the algorithm at the end stage we attach copies of K_2 to all the pendant vertices. Even if we replace K2 with a path of any length, the resulting tree will still be equitable irregular.

5. EQUITABLE IRREGULARITY OF SOME CLASSES OF TREES

Definition 5.1. A caterpillar [7] is a tree in which the removal of all pendant vertices will result in a path. The path is called the spine of the caterpillar.

Theorem 5.1. A caterpillar is equitable irregular if and only if it is a path.

Proof. Let T be a caterpillar which is a path. Assigning weight 1 to all edges clearly results in an equitable irregular edge weighting for T. Now let T be a caterpillar which is not a path. To prove T is not equitable irregular. Let u be a vertex not in the spine of the caterpillar. Then u is a pendant vertex. Let v be a vertex in the spine of the caterpillar adjacent to u.

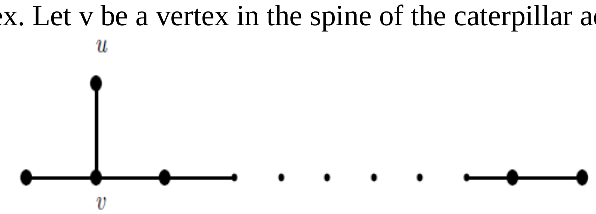


Figure 13: A caterpillar tree

Suppose Ø(u) = a. Then S_(u) = a. Since v is adjacent to at least two other vertices, we must have $S_Ø$ (v) ≥ a + 2. So, |$S_Ø$ (u) - $S_Ø$ (v)|≥ 2. Hence T is not equitable irregular.

Definition 5.2. A tree T is said to be a **Lobster** [9] if after deleting all its pendant vertices we get a caterpillar. The path of the caterpillar tree corresponding to T is called the path of T.

Now we make a small observation on the equitable irregularity of some lobsters.

Observation 5.1. If T is an equitable irregular lobster tree of order at most 8, then T is either a path or a one subdivision of a star.

Proof. Proof is a simple exercise and can be verified easily.

Sahul Hamid and Ashok Kumar[5] have found a necessary condition for any graph to be equitable irregular discussed in Theorem 2.1. In particular this theorem gives a necessary condition for a lobster tree to be equitable irregular. Here we give a sufficient condition for a lobster to be equitable irregular.

Proposition 5.1. A lobster tree T possessing the following properties

1. The degree of any pair of adjacent vertices in the central path differ by at most 1.
2. All support vertices are of degree 2. is equitable irregular with S_e(T) = Δ(T)- 2.

Proof. Let T be a lobster tree having the given properties. Define an edge weighting Ø on T as follows. For any edge *uv* in T, Ø(uv) = 1, if uv is on the central path of T or uv is adjacent to a vertex on the central path; Ø(uv) = deg(w)-2, if u or v is adjacent to w, a vertex on the central path. This gives an equitable irregular edge weighting of T with equitable irregular strength Δ(T)- 2.

Definition 5.3. [8] A rooted tree is called an *m*-ary tree if every internal vertex has no more than *m* children. An *m*-ary tree with m = 2 is called a **binary tree.**

Theorem 5.2. A binary tree is equitable irregular if and only if all support vertices have degree 2.

Proof. If T is a binary tree that is equitable irregular, then clearly all its support vertices have degree 2. Conversely let T be a binary tree with all its support vertices of degree 2. Let vi be a vertex in the i^{th} and vi+1 be a vertex in the $(i+1)^{th}$ level of T. Then by definition, the degrees of v_i and v_{i+1} are either 2 or 3. If T has n levels, by assumption v_{n-1} has degree 2 and v_n has degree 1. Hence the degrees of all pairs of adjacent vertices differ by at most 1. So, T is equitable irregular.

6. CONCLUSION AND SCOPE OF STUDY

In this paper, we carried out a study on the equitable irregularity of some graphs. We obtained some equitable irregular graphs by attaching odd length cycles. The case while attaching even length cycles is still unsolved. We obtained a sufficient condition for equitable irregularity of graphs. We have checked this property for a few classes of trees. We have identified a way of constructing more trees that are equitable irregular with base as stars. We have characterized Caterpillars for this property and found some results in lobsters. Still there is a wide scope in studying various other families and get some interesting properties. We have found the characterization of lobsters for this property with some restrictions. A necessary and sufficient condition for equitable irregularity of graphs is still not found.

REFERENCES

1. G. Chartrand *et al*., "Irregular networks," *Congr. Numer,* vol. 64, pp. 197 – 210, (1988).
2. G. Chartrand and P. Zhang, "*Edge Colorings of Graphs*," in Chromatic Graph Theory, K.H. Rosen, Ed. New York: CRC Press, pp. 249-262 (2009).
3. J. Gallian, "A dynamic survey of graph labelling," *Electron. J. Combinator*., vol.16, (2013).
4. M. Karonski *et al.,* "Edge weights and vertex colours," *J. Combinator. Theory* B, vol. 91, pp. 151-157, (2004).
5. I.S. Hamid and S.A. Kumar, "Equitable irregular edge-weighting of graphs," *SUT J. of Math*., vol.46, no. 1, pp. 79-91.
6. A. Rosa, "On certain valuations of the vertices of a graph," at International Symposium., Rome, (July 1966).
7. D.B. West, "*Line Graphs and Edge Colorings*," in Introduction to graph theory, 2nd ed, pearson, pp. 273-281 (2001).
8. K.H Rosen, "Trees," in *Discrete mathematics and its applications*, 7th ed., Mc Graw Hill, pp.745-757 (2012).
9. E. Weisstein "Lobster Graph, A Wolfram Web Resource, [Online]. Available: http://mathworld.wolfram.com/LobsterGraph.html
10. E. Weisstein "Ladder Graph,"A Wolfram Web Resource, [Online]. Available: http://mathworld.wolfram.com/LadderGraph.html