Structural Characterization of Super Strongly Perfect Graphs on Trees

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ABSTRACT

A Graph G is Super Strongly Perfect Graph if every induced sub graph H of G possesses a minimal dominating set that meets all the maximal cliques of H. In this paper, we have given a characterization of Super Strongly Perfect graphs using odd cycles. Using this characterization we have characterized the Super Strongly Perfect graphs on Trees. We have given the relationship between diameter $diam(G)$, domination number $\gamma$ and co-domination number $\gamma'$. Also we have analysed the structure of Super Strongly Perfect Graph on Spider graphs, Wounded Spider graphs and Galaxy graphs.

Keywords: Super Strongly Perfect Graph, Minimal Dominating Set, Domination and Co-Domination numbers and Tree.

1. INTRODUCTION

Graph theory is becoming increasingly significant as it is applied to other areas of mathematics, science and technology. The powerful combinatorial methods found in graph theory have also been used to prove fundamental results in other areas of pure mathematics. Super Strongly Perfect graph is defined by B. D. Acharya and its Characterization has been given as an open problem in. No work has been done previously under this topic. We have proved a characterization of Super Strongly Perfect Graph. Using this Super Strongly Perfect Graph, we have given the result of characterization of both Strongly Perfect graphs and Perfect graphs. Because
of this characterization, we can apply all applications of Perfect graphs to Super Strongly Perfect graphs.

In an undirected graph, a cycle is simply a set of edges with respect to which every vertex has even degree. Cycles in graphs play an important role in many applications, e.g., network analysis, periodic scheduling and surface reconstruction. A tree is a mathematical structure that can be viewed as either a graph or as a data structure. The two views are equivalent, since a tree data structure contains not only a set of elements, but also connections between elements, giving a tree graph. Trees were first studied by Cayley\textsuperscript{1}. A tree is a special kind of graph follow a particular set of rules. A tree is a set of straight line segments connected at their ends containing no closed loops (cycles). In other words, it is a simple, undirected, connected, acyclic graph. A tree does not have a specific direction. Depending on how it is to be used, the tree may branch outward while going upward (like a real tree - the growing kind), or it can branch down like the roots of a real tree.

2. BASIC CONCEPTS

In this paper, graphs are finite and simple, that is, they have no loops or multiple edges. Let $G = (V, E)$ be a graph. A clique in $G$ is a set $X \subseteq V (G)$ of pair wise adjacent vertices. A subset $D$ of $V (G)$ is called a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in $D$. A subset $S$ of $V$ is said to be a minimal dominating set if $S - \{u\}$ is not a dominating set for any $u \in S$. The domination number $\gamma (G)$ of $G$ is the smallest size of a dominating set of $G$. The domination number of its complement $\bar{G}$ is called the co-domination number of $G$ and is denoted by $\gamma (\bar{G})$ or simply $\bar{\gamma}$. A shortest uv path of a connected graph $G$ is often called a geodesic. The diameter denoted by $\text{diam}(G)$ is the length of any longest geodesic. A vertex $v$ of degree zero in $G$ is called an isolated vertex of $G$. The minimum degree of a graph is denoted by $\delta(G)$. The star graph on $n$ vertices with one vertex having vertex degree $n - 1$ and the other $n - 1$ vertices having vertex degree 1. That is star is a complete bipartite graph $K_{1,n-1}$.

3. OVERVIEW OF THE PAPER

In this paper, we have characterized the Super Strongly Perfect graphs using odd cycles. Using this characterization we have also characterized the Super Strongly Perfect graphs on Trees. We have found the relationship between diameter $\text{diam}(G)$, domination number $\gamma$ and co-domination number $\bar{\gamma}$ of super strongly perfect graph in trees. Also we have discussed the tree structures of Spider graphs, Wounded Spider graphs and Galaxy graphs.

3.1. Super Strongly Perfect Graph

A Graph $G = (V, E)$ is Super Strongly Perfect if every induced sub graph $H$ of $G$ possesses a minimal dominating set that meets all the maximal complete sub graphs of $H$.

Example 1

![Figure 1: Super Strongly Perfect Graph](image)
Here, \{5, 8\} is a minimal dominating set which meets all maximal cliques of G.

**Example 2**

![Figure 2: Non-Super Strongly Perfect Graph](image)

Here, \{1, 3, 5, 7\} is a minimal dominating set which does not meet all maximal cliques of G.

### 4. CYCLE GRAPH

A Cycle graph or Circular graph is a graph that consists of a single cycle or in other words some number of vertices connected in a closed chain and it is denoted by \(C_n\). The number of vertices in \(C_n\) equals the number of edges. The cycle graph with even number of vertices is called an even cycle and the cycle graph with odd number of vertices is called an odd cycle.

In cycle families, we have two dimensions, based on the number of vertices. It has the peculiar behavior according to their parameter. Even cycles follow the family of Super Strongly Perfect Graph whereas the odd cycles don’t.

#### 4.1. Theorem [6]

Let \(G = (V, E)\) be a Cycle with odd number of vertices, \(n \geq 5\), then \(G\) is Non-Super Strongly Perfect.

#### 4.2. Theorem [6]

Let \(G = (V, E)\) be a graph with number of vertices \(n\), where \(n \geq 5\). If \(G\) contains an odd cycle as an induced subgraph, then \(G\) is Non-Super Strongly Perfect.

### 4.3. Theorem [2]

Let \(G\) be a graph. If for no odd \(n \geq 5\), \(C_n\) or \(\overline{C_n}\) is an induced subgraph of \(G\), then \(\omega(G) = \chi(G)\).

#### 4.4. Theorem

Let \(G\) be a graph. Then \(G\) is Super Strongly Perfect if and only if it does not contain an odd cycle \(C_n\), \(n \geq 5\) as an induced subgraph.

**Proof:**

Let \(G\) be a graph.

Assume \(G\) does not contain an odd cycle \(C_n\), \(n \geq 5\) as an induced subgraph.

by the theorem 4.3, \(\chi(G) = \omega(G) = k\) (say)

\(\Rightarrow\) There exists \(k\)-partition sets \(V_1, V_2, V_3, \ldots, V_k\) such that \(V_1 \cup V_2 \cup V_3 \cup \ldots \cup V_k = V\).

Let \(S = \{v_1, v_2, v_3, \ldots, v_k\}\), \(v_i \in V_i\), \(i = 1, 2, \ldots, k\) be a minimal dominating set of \(G\).

To prove \(G\) is Super Strongly Perfect.

If \(G\) is not super strongly perfect, then there exists at least one maximal clique \(C\) which is not met by \(S\).

Any vertex of the maximal clique does not belong to \(S\).

\(\Rightarrow\) there exists a vertex \(v \notin S\) but \(v \in C\).

\(\Rightarrow v \notin V_i\), \(i = 1, 2, 3, \ldots, k\).

\(\Rightarrow v\) should be coloured with some \((k+1)\)th colour.

\(\Rightarrow \omega(G) = k + 1\). This is a contradiction to our assumption.
Hence G is Super Strongly Perfect.
Conversely assume that G is Super Strongly Perfect.
To prove, G does not contain an odd cycle C_n, n ≥ 5 as an induced sub graph.
If G contains an odd cycle C_n, n ≥ 5 as an induced sub graph,
Then by the theorem 4.2, G is Non - Super Strongly Perfect.
This is a contradiction to the assumption.
Hence G does not contain an odd cycle C_n ≥ 5 as an induced sub graph.
Hence the theorem is proved.

5. TREE

An acyclic graph is a graph which contains no cycle. A tree is a connected acyclic graph. Every tree on n vertices has exactly n-1 edges. Any two vertices of a tree are connected by exactly one path.

The above specified parameter will behave positively in tree structure.

Example

![Figure 3: Tree](image.png)

5.1. Theorem

Every Tree is Super Strongly Perfect.

Proof:
Let G be a Tree.

⇒ G does not contain an odd cycle as an induced sub graph.
Now, by the theorem 4.4, G is Super Strongly Perfect.
Hence every Tree is Super Strongly Perfect.

5.2. Theorem

Let G be a tree which is Super Strongly Perfect then γ(G) = 1 if and only if diam(G) = 2.

Proof:

Let G be a tree which is Super Strongly Perfect.
Assume that γ(G) = 1.
⇒ There exists a vertex v ∈ G which is adjacent to all the remaining vertices in G.
To prove diam(G) = 2.
Suppose diam(G) ≠ 2.
⇒ either diam(G) < 2 or diam(G) > 2.
Case (i): If diam(G) < 2
then diam(G) = 1
⇒ G is complete
which is a contradiction, since G is a tree.

Case (ii): If diam(G) > 2
Then there exists at least two vertices a and b with diam(a, b) ≥ 3.
⇒ There does not exist a vertex in G which is adjacent to both a and b.
⇒ γ(G) > 1.
Which is a contradiction to the assumption, hence diam(G) = 2.
Conversely assume that diam(G) = 2.
To prove γ(G) = 1.
Suppose γ(G) ≠ 1,
⇒ γ(G) ≥ 2, then there does not exist a vertex in G which is adjacent to all
the remaining vertices.
⇒ There exists at least two vertices \( a \) and \( b \) such that \( \text{diam}(a, b) \geq 3 \).
⇒ \( \text{diam}(G) \geq 3 \).
Which is a contradiction to the assumption, hence \( \gamma(G) = 1 \).

5.3. Theorem

Let \( G \) be a tree which is Super Strongly Perfect with \( \gamma(G) = 1 \), then \( \gamma(\overline{G}) = 2 \) if and only if \( \text{diam}(\overline{G}) \) is not defined.

Proof:

Let \( G \) be a tree which is Super Strongly Perfect with \( \gamma(G) = 1 \).
Assume that \( \gamma(\overline{G}) = 2 \).
To prove \( \text{diam}(\overline{G}) \) is not defined.
Since \( \gamma(G) = 1 \), there exists a vertex \( v \in G \) which is adjacent to all the vertices in \( G \).
⇒ \( v \) is an isolated vertex in \( \overline{G} \).
Since \( \gamma(\overline{G}) = 2 \) and \( v \) is an isolated vertex in \( \overline{G} \), hence \( v \) must be one of the vertex of the minimum dominating set of \( \overline{G} \).
Let \( u \in \overline{G} \) such that \( \{u, v\} \) be a minimum dominating set of \( \overline{G} \).
⇒ \( \text{diam}(u, v) \) is not defined in \( \overline{G} \), since \( v \) is an isolated vertex in \( \overline{G} \).
⇒ \( \text{diam}(\overline{G}) \) is not defined.
Conversely assume that \( \gamma(G) = 1 \) and \( \text{diam}(\overline{G}) \) is not defined.
To prove \( \gamma(\overline{G}) = 2 \).
Since \( \text{diam}(\overline{G}) \) is not defined, there exists a vertex \( v \in \overline{G} \) which is an isolated vertex such that \( \text{diam}(u, v) \) is not defined for some \( u \in \overline{G} \).
Since \( \gamma(G) = 1 \), \( \{v\} \) will be the minimum dominating set of \( G \).
Since \( G \) is a tree, there exists at least one pendent vertex \( u^l \) incident with a pendent edge, let it be \( e = u^l v^l \in G \).
⇒ \( u^l \) is a pendent vertex in \( G \), it is adjacent with all the remaining vertices except \( v^l \) in \( \overline{G} \).
⇒ There exists a vertex \( u^l \) which is adjacent with all the remaining vertices in \( \overline{G} \) and \( v \) is an isolated vertex in \( \overline{G} \).
⇒ \( \{u^l, v\} \) is a minimum dominating set of \( \overline{G} \).
⇒ \( \gamma(\overline{G}) = 2 \).
Hence proved.

5.4. Theorem

Let \( G \) be a tree which is Super Strongly Perfect, then \( \gamma(G) > 1 \) if and only if \( \text{diam}(G) \geq 3 \).

Proof:

Let \( G \) be a tree which is Super Strongly Perfect.
Assume that \( \text{diam}(G) \geq 3 \).
To prove \( \gamma(G) > 1 \).
By the assumption, \( \text{diam}(G) \geq 3 \).
Then there exists at least two vertices \( u \) and \( v \) of \( G \) such that \( \text{diam}(u, v) \geq 3 \).
⇒ There does not exist a vertex in \( G \) which is adjacent to both \( u \) and \( v \).
⇒ \( \gamma(G) > 1 \).
Conversely assume that \( \gamma(G) > 1 \).
⇒ \( \gamma(G) \neq 1 \).
By the theorem 5.2, \( \text{diam}(G) \neq 2 \).
⇒ either \( \text{diam}(G) < 2 \) or \( \text{diam}(G) > 2 \).

Case (i): If \( \text{diam}(G) < 2 \)
then \( \text{diam}(G) = 1 \)
⇒ \( G \) is complete
which is a contradiction, since \( G \) is a tree.
⇒ \( \text{diam}(G) > 2 \)
⇒ \( \text{diam}(G) \geq 3 \).
Hence proved.
5.5. Theorem

Let G be a tree which is Super Strongly Perfect with \( \text{diam} (G) = 2 \) then \( \gamma (G) = \delta (G) \).

Proof:

Let G be a tree which is Super Strongly Perfect with \( \text{diam} (G) = 2 \).
Since G is a tree, \( \delta (G) = 1 \).
To prove \( \gamma (G) = \delta (G) \).
It is enough to prove \( \gamma (G) = 1 \).
Suppose \( \gamma (G) > 1 \)
Then by theorem 5.4, \( \text{diam} (G) \geq 3 \).
Which is a contradiction to the hypothesis.
Hence our assumption is wrong.
\( \Rightarrow \gamma (G) = 1 \).
\( \Rightarrow \gamma (G) = \delta (G) \).
Hence proved.

5.6. Theorem

Let G be a tree which is Super Strongly Perfect, then \( \text{diam} (G) \) and \( \text{diam} (\tilde{G}) \) cannot be 1.

Proof:

Let G be a tree which is Super Strongly Perfect.
To prove \( \text{diam} (G) \neq 1 \) and \( \text{diam} (\tilde{G}) \neq 1 \).
Suppose \( \text{diam} (G) = 1 \).
\( \Rightarrow \) All the vertices are pairwise adjacent in G.
\( \Rightarrow \) G is complete, which is a contradiction to the hypothesis.
Hence our assumption is wrong.
\( \Rightarrow \text{diam} (G) \neq 1 \).
Now to prove \( \text{diam} (\tilde{G}) \neq 1 \).
Suppose \( \text{diam} (\tilde{G}) = 1 \).

5.7. Proposition

Let G be a tree which is Super Strongly Perfect, then \( \text{diam} (G) \geq 3 \) if and only if \( \gamma (\tilde{G}) = 2 \).

5.8. Proposition

Let G be a tree which is Super Strongly Perfect with number of vertices, then \( \text{diam} (\tilde{G}) \leq 3 \) if and only if \( \gamma (\tilde{G}) = 2 \).

In the following sections, we are going to discuss some of the tree structure such as Spider, Wounded Spider and Galaxy graphs.

6. SPIDER

A Spider is the graph formed by subdividing all the edges of a star \( K_{1, t} \), for \( t \geq 0 \), exactly once. A subdivision of an edge uv is obtained by removing edge uv, adding a new vertex w, and adding edges uw and vw.
A Spider is a tree with one vertex of degree at least 3 and all others with degree at most 2. A Spider is a tree with at most one vertex of degree more than two, called the center of Spider. A leg of a Spider is a path from the center to a vertex of degree one. Thus, a star with \( k \) legs is a Spider of \( k \) legs, each of length 1.
Example

Figure 4: Spider

6.1. Theorem

Every Spider is Super Strongly Perfect.

Proof:

Let G be a Spider.
⇒ G does not contain an odd cycle as an induced sub graph.
Now, by the theorem 4.4, G is Super Strongly Perfect.
Hence every Spider is Super Strongly Perfect.

7. WOUNDED SPIDER

A Wounded Spider is the graph formed by subdividing at most t -1 of the edges of a star K_{1,t} for t ≥ 0. A subdivision of an edge uv is obtained by removing edge uv, adding a new vertex w, and adding edges uw and vw. In a wounded spider, the vertex of degree t is called the head vertex, and the vertices that are distance two from the head vertex are the foot vertices. Both vertices in P_2 are considered to be head vertices, and in the case of P_4, both end vertices are considered foot vertices whereas the two central vertices are head vertices.

7.1. Theorem

Every Wounded Spider is Super Strongly Perfect.

Proof:

Let G be a Wounded Spider.
⇒ G does not contain an odd cycle as an induced sub graph.
Now, by the theorem 4.4, G is Super Strongly Perfect.
Hence every Wounded Spider is Super Strongly Perfect.

8. GALAXY

A forest is a graph with tree components. A Galaxy is a forest in which each component is a star. A star is an arborescence in which the root dominates all the other vertices. A galaxy is a vertex-disjoint union of stars^3.

Example

Figure 6: Galaxy
8.1. Theorem

Every Galaxy is Super Strongly Perfect.

Proof:

Let G be a Galaxy.
⇒ G does not contain an odd cycle as an induced sub graph.
Now, by the theorem 4.4, G is Super Strongly Perfect.
Hence every Galaxy is Super Strongly Perfect.

9. CONCLUSION

We have given the characterization of Super Strongly Perfect graphs using odd cycles. Also, we have given the characterization of Super Strongly Perfect graphs on Trees. Moreover, we have investigated the structure of Spider graphs, Wounded Spider graphs and Galaxy graphs. This investigation can be applicable for the well known architectures like hypercube graphs, butterfly graphs, benes graphs, chordal graphs etc.

REFERENCES