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Some Fixed Point Results in Complete Menger Spaces with Occasionally Weakly Compatible Conditions

Manoj Kumar Shukla[1], M. S Chauhan[2], Sheeba Khan[3] and Akanchha Singh[4]

Institute For Excellence In Higher Education, Bhopal (M.P) INDIA.
email:[1]manojshukla_10@yahoo.com, dr.msc0401@gmail.com[2]
[3]sheebakhan274@gmail.com. akanchha369singh@gmail.com

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ABSTRACT

In this present paper we have established some fixed point results by utilizing the notion of Occasionally Weakly Compatibility in complete Menger Spaces.

Keywords: Occasionally Weakly Compatible, Complete Menger Space.

1. INTRODUCTION

We can generalized the Metric space. An important generalization of metric spacr is Menger Space introduced in 1942 by Menger[4] taking distribution functions instead of non negative real numbers as values of the metric, the notion of probabilistic metric space correspond to situation when we do not know exactly the distance between the two points but we know probabilities of possible values of this distance. A probabilistic generalization of metric space appears to be interest in the investigation of physical quantities and physiological threshold. It is also a fundamental importance in probabilistic functional analysis. Schweizer and Sklar[7] studied this concept and then the important development of Menger space theory was due to Sehgal[9] and Bharucha-Reid[1]. The development of fixed point theory in PM- spaces was due to Schweizer and Sklar[8]. Further Sehgal[9] initiated the study of contraction mapping theorems in PM-spaces. Subsequently, Ciric *et al.*[2], Pant and Kumar[5] did lot of work in this field.

In this paper we have proved some fixed point results in Menger Spaces satisfying an Occasionally Weakly Compatible mapping which is an extension of the recent work of Shukla.M, Dubey.R, Patel.S.K.

The study of fixed points for non-compatible mappings can be extended to the class of non-expansive or Lipschitz mapping pairs even without continuity of the mappings involved.

Besides the above we choose to utilize the notion of occasionally weakly compatible to prove our result in Menger space, which is a wider and suitable framework in many situations (1942). Here one may observe that we need not improve the completeness requirements of the space or the containment of the ranges of the involved mappings. Our result complement comparable results of Ciric *et al*.[3], Sedghi *et al*.[10] & Singh *et al*.[11].

The following definition and lemma are required before jump over to our main result:

2. PRELIMINARIES

Definition 2.1: Let X be a non empty set and L denote the set of all distribution functions. A probabilistic metric space is an ordered pair (X, F) where $F: X * X \to L$. we shall denote the distribution function by F (p, q) or $F_{p,q}$; p, q ϵ X and F(p, q, x) will represent the value of F (p, q) at x ϵ R. the function $F_{p,q}$ is assumed to satisfy the following conditions :

1. $F_{p,q}(t) = 1 \ \forall\ t > 0$ if and if p = q
2. $F_{p,q}(0) = 0$ for every p,q ϵ X
3. $F_{p,q}(t) = F_{q,p}(t)$ for every p, q ϵ X
4. If $F_{p,q}(t) = 1$ and $F_{q,r}(s) = 1$ it follows that $F_{q\,r}(t+s) = 1 \ \forall$ p,q,r ϵX and $t, s \geq 0$.

In metric space (X ,d) , the metric d induces a mapping $F: X * X \to L$ such that $F_{p,q}(t) = H(t - d(p,q))$ for all p, q ϵ X and t ϵ R , where His the distribution function defined as

$$H(x) = \begin{cases} 0, \text{if } x \leq 0 \\ 1, \text{if } x > 0 \end{cases}$$

Definition 2.2: A mapping $\Delta: [0,1] * [0,1] \to [0,1]$ is called t- norm if the following conditions are satisfied

(1) $\Delta(a, 1) = a$ for all a ϵ [0, 1], $\Delta(0,0) = 0$,

(2) $\Delta(a,b) = \Delta(b,a)$

(3) $\Delta(c,d) \leq \Delta(a,b)$ for $c \geq a$, $d \geq b$, and

(4) $\Delta(\Delta(c,d),c) = \Delta(a,\Delta(b,c))$ for all a, b, c ϵ [0,1]

Example: (1) $\Delta(a,b) = a\,b$, **(2)** $\Delta(a,b) = \min(a,b)$ **(3)** $\Delta(a,b) = \max(a+b-1;0)$

Definition 2.3: A Menger probabilistic space is a triplet (X, F, Δ) where (X, F) is a PM-space and Δ is a t- norm with the following condition

$F_{p,r}(\mathbf{t+s}) \geq \Delta(F_{p,r}(\mathbf{t}), F_{p,r}(\mathbf{s}))$ for all p, q, r ϵ X and $t, s \geq 0$.

The above inequality is called Menger's triangle inequality.

Definition 2.4: Let (X, F,Δ) be a Menger Space. If u ϵ X, $\varepsilon > 0$, $\lambda\ \epsilon\ (0,1)$, then an (ε,λ) neighborhood of u, called $U_u(\varepsilon,\lambda)$ is defined as

$U_u(\varepsilon,\lambda) - \{v\ \epsilon X; F_{u,v}(\varepsilon) > 1 - \lambda\}$.

An (ε,λ) –topology in X is the topology induced by the family

$\{U_u(\varepsilon,\lambda)\ ;\ u\ \epsilon\ X,\ \varepsilon > 0,\ \lambda\ \epsilon\ (0,1)\}$ of neighborhood. If t is continuous, then Menger Space (X,F,Δ) is a Hausdroff space in the (ε,λ)- topology.

Definition 2.5: A sequence $\{x_n\}$ in (X, F,Δ) is said to be convergent to a point x ϵ X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon,\lambda)$ such that $F_{x_n,X}(\varepsilon) \to 1 - \lambda \ \forall\ n \geq N(\varepsilon,\lambda)$.

Definition 2.6: A sequence $\{x_n\}$ in (X,F,Δ) is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N=N(\varepsilon,\lambda)$ such that $F_{x_n, x_m}(\varepsilon) \rightarrow 1-\lambda \ \forall\ n, m \geq N \ (\varepsilon,\lambda)$.

Definition 2.7: A Menger Space (X, F,Δ) with the continuous t- norm is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.8: Two self mappingf and g of a probabilistic metric space (X,F,Δ) are said to be point wise R- weakly commuting if given $x \in X$, there exists $R > 0$ such that
$F_{fgx, gfx}(t) \geq F_{fx, gx}(t/R)$ for $t > 0$.
And T and R-weak commutative of type (A_g) if $F_{ffx, gfx}(t) \geq F_{fx, gx}(t/R)$

Definition 2.9: Two self maps f and g of a set X are OWC iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

Definition 2.10: A function $\emptyset$: $[0,\infty)$ is said to be a $\emptyset$- function if it satisfies the following conditions:

I. $\emptyset(t) = 0$ if and only if $t = 0$,
II. $\emptyset(t)$ is strictly increasing and $\emptyset(t) \rightarrow \infty$ as $t \rightarrow \infty$
III. $\emptyset(t)$ is left continuous in$(0,\infty)$ and
IV. $\emptyset$ is continuous at 0,

An altering distance function with the additional property that $h(t) \rightarrow \infty$ as $t \rightarrow \infty$ as generates a $\emptyset$ function in the following way:

$$\emptyset(t) = \begin{cases} \sup\{s: h(s) < t \ if\ t > 0 \\ \quad 0 \qquad if\ t = 0 \end{cases}$$

It can be easily seen that $\emptyset$ is a function $\emptyset$- function.

Lemma 2.1: Let (X, F, Δ) be a menger space, A and B are OWC self maps of X. If A and B have a unique point of coincidence, w = Ax = Bx, then w is the unique common fixed point of A and B.

Proof: since A and B are occasionally weakly compatible, there exists a point $x \in X$ such that Ax = Bx = w and ABx = BAx. Thus, AAx = ABx =BAx, which says that Ax is also a point of coincidence of A and B. since the point of coincidence w = Ax is unique by hypothesis, BAx = AAx = Ax, and w = Ax is a common fixed point of A and B. moreover, if z is any common fixed point of A and B, then
Z = Az = Bz =w by the uniqueness of the point of coincidence.
Following theorem is already in pipeline proved by our co researches under the guidance of our supervisor Shukla Manoj Kumar.

Theorem 2.1 Let (X, F, Δ) be a complete Menger Space with continuous t-norm Δ and $\Delta(t, t) \geq t$ for all $t \in [0, 1]$ and let p, q, f and g be self mapping of X. Let pairs {p, f} and {q, g} be OWC. If there exists $k \in (0, 1)$ such that

$$F_{px,qy}(kt) \geq \begin{matrix} \Delta(\Delta F_{fx,gy}(t), \Delta(F_{px,fx}(t), \Delta F_{qy,gy}(t), \Delta F_{px,gy}(t), \Delta F_{fx,qy}(t), \\ \Delta(\frac{F_{qy,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2}))))) \end{matrix} \quad \ldots..2.4.1$$

for all x, y ϵ X and t > 0, then there exists a unique common fixed point of p, f, q, and g.

Theorem 2.2: Let (X, F, Δ) be a complete Menger space with continuous t-norm Δ and $\Delta(t, t) \geq t$ for all $t \in [0, 1]$. Then continuous self mappings f and g of X have a common fixed point in X if and only if there exists a self mapping p of X such that the following conditions are satisfied.

(3.1.1) $p(X) \subset g(X) \cap f(X)$

(3.1.2) The pairs $\{p, f,\}$ and $\{p, g\}$ are weakly compatible,

(3.1.3) There exist a point $h \in (0,1)$ such that for every $x, y, \in, X$ and $t > 0$.

$$F_{px,Py}(kt) \geq \begin{matrix} \Delta(\Delta F_{Px,gy}(t), \Delta(F_{px,fx}(t), \Delta F_{Py,gy}(t), \Delta F_{px,gy}(t), \Delta F_{fx,Py}(t), \\ \Delta(\frac{F_{Py,gy}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,gy}(t)+F_{px,gy}(t)}{2})))) \end{matrix}$$

for all $x, y, \in, X$ and $t > 0$, the *p, f, and g* have a unique common fixed point.

3. MAIN RESULTS

Theorem3.1: Let (X, F, Δ) be a complete Menger Space and let p and f be self mappings of X, Let p and f be OWC. If there exits $k \in (0, 1)$ for all $x,y \in X$ *and* $t > 0$

$$F_{fx,fy}(kt) \geq \begin{bmatrix} \alpha Fpx,py(t) + \beta min\{Fpx,fx(t), Ffy,py(t), Fpx,fy(t) \\ +\gamma \min\{Ffx,py(t), \frac{F_{fy,py}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,py}(t)+F_{px,fy}(t)}{2}\} \end{bmatrix} \quad 3.1.1$$

For all $x, y \in X$ and $t > 0$, where $\alpha, \beta, \gamma > 0$, $\alpha+\beta+\gamma > 1$. Then there exists a unique common fixed point of *p and f.*

Proof: since the pairs $\{p, f\}$ and $\{q, g\}$ are (OWC), there exists points $x, y \in X$ such that $px = fx$ and $qy = gy$. We claim that $px = qy$. By inequality 3.1.1 we have

$$F_{fx,fy}(kt) \geq \begin{bmatrix} \alpha Fpx,py(t) + \beta min\{Fpx,fx(t), Ffy,py(t), Fpx,fy(t) \\ +\gamma \min\{Ffx,py(t), \frac{F_{fy,py}(t)+F_{fx,px}(t)}{2}, \frac{F_{fx,py}(t)+F_{px,fy}(t)}{2}\} \end{bmatrix}$$

$$F_{fx,fy}(kt) \geq \begin{bmatrix} \alpha Ffx,fy(t) + \beta min\{Fpx,px(t), Ffy,fy(t), Ffx,fy(t) \\ +\gamma \min\{Ffx,fy(t), \frac{F_{fy,fy}(t)+F_{fx,fx}(t)}{2}, \frac{F_{fx,fy}(t)+F_{fx,fy}(t)}{2}\} \end{bmatrix}$$

$$= (\alpha+\beta+\gamma) F_{fx,fy}(t)$$

A contradiction, since $\alpha+\beta+\gamma > 1$. Therefore $fx = fy$. Therefore $px = py$ and px is unique. From lemma 2.1 p and f have a unique fixed point.

Theorem3.2: Let (X, F, Δ) be a complete Menger Space and let *p, q, fand g* be self mappings of X. Let pairs $\{p, f\}$*and* $\{q, g\}$ be OWC. If there exits $k \in (0, 1)$ such that

$$F_{px,,qy}(kt) \geq \begin{bmatrix} \alpha F_{px,fx}(t) + bF_{px,gy}(t) + cF_{fx,qy}(t) \\ +dmin\{F_{qy,gy}(kt), F_{fx,gy}(kt)\} \end{bmatrix} \ldots\ldots.3.2.1$$

For all x , y ∈ X and for all t > 0, where 0 < a, b, c, d < 1 such that a + b + c + d = 1; then p, f, q and g have a unique common fixed point in X.

Proof: since the pairs *{p, f} and {q, g}* are (OWC), there exists points $x, y \in X$ such that $px = fx$ and qy = gy. We claim that $px = qy$. By inequality 3.2.1 we have

$$F_{px,,qy}(kt) \geq \begin{bmatrix} \alpha F_{px,fx}(t) + bF_{px,gy}(t) + cF_{fx,qy}(t) \\ +d\min\{F_{qy,gy}(kt), F_{fx,gy}(kt)\} \end{bmatrix} \ldots\ldots.3.2.1$$

$$= \begin{bmatrix} \alpha F_{px,px}(t) + bF_{px,gy}(t) + cF_{px,qy}(t) \\ +d\min\{F_{qy,qy}(kt), F_{px,qy}(kt)\} \end{bmatrix}$$

$$= a + (b+c)\, F_{px,qy}(t) + d\min\{1, F_{px,qy}(kt)\}$$

$$= a + (b+c+d)\, F_{px,qy}(t)$$

$$F_{px,qy}(kt) \geq \frac{a}{1-b-c-d}$$

$$F_{px,qy}(kt) \geq 1.$$

Thus, we have $px = qy$. Therefore, $px = qy = fx = gy$.

Suppose that there is another point z such that $pz = fz$ then we have $pz = fz = qy = gy$, so $px = pz$ *and* $w = px = fx$ is the unique point of coincidence of *p and f.*

By lemma (2.1) w is the only common fixed point of p and f. similarly there is a unique point $z \in X$ *such that* $z = qz = gz$.

Suppose $w \neq z$ taking x = w, y = z in inequality 3.3.1, we get

$$F_{w,z}(kt) = F_{pw,qz}(kt)$$

$$\geq \begin{bmatrix} \alpha F_{pw,fw}(t) + bF_{pw,gz}(t) + cF_{fw,qz}(t) \\ +d\min\{F_{qz,gz}(kt), F_{fw,gz}(kt)\} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha F_{w,w}(t) + bF_{w,z}(t) + cF_{w,z}(t) \\ +d\min\{F_{z,z}(kt), F_{w,z}(kt)\} \end{bmatrix}$$

$$= a + (b+c)\, F_{w,z}(t) + d\min\{1, F_{w,z}(kt)\}$$

$$= a + (b+c+d)\, F_{w,z}(t)$$

$$F_{w,z}(kt) \geq \frac{a}{1-b-c-d}$$

$$F_{w,z}(kt) \geq 1.$$

Thus, we have w=z. that is w is the unique common fixed point of p, f, q and g in X.

Example: let X = [0,1] with metric d defined by d(x,y) = |x-y| and for each $t \in [0,1]$ define

$$F_{x,y}(t) = \begin{cases} e^{-|x-\frac{y}{t},|} \\ 0, \end{cases} \quad \text{if } t>0; \text{ if } t = 0.$$

$$P(x) = \begin{cases} x, & if\ 0 \leq x \leq \frac{1}{2} \\ 1. & if\ \frac{1}{2} < x \leq 1. \end{cases} \qquad f(x) = \begin{cases} \frac{1}{2}, & if\ 0 \leq x \leq \frac{1}{2} \\ 0. & if\ \frac{1}{2} < x \leq 1. \end{cases}$$

$$q(x) = \begin{cases} \frac{1}{2}, & if\ 0 \leq x \leq \frac{1}{2}; \\ 1. & if\ \frac{1}{2} < x \leq 1. \end{cases} \qquad g(x) = \begin{cases} \frac{1}{2}, & if\ 0 \leq x \leq \frac{1}{2}; \\ \frac{x}{4}. & if\ \frac{1}{2} < x \leq 1. \end{cases}$$

Then p, q, f and g satisfy all the conditions of theorem 3.1 for $k \in (0, 1)$ with respect to the distribution function $F_{x,y}$.

Thus $(\frac{1}{2})$ is the unique common fixed point of p, q, f and g and also we see that the mappings p, q, f and g are discontinuous at $(\frac{1}{2})$.

Remark: in all the above theorem p, q, r, s all may be set valued functions.

4. CONCLUSION

Occasionally Weakly Compatible (OWC) mappings always commute at coincidence point of mappings and we have find this coincidence point as common fixed point of mappings. These OWC mappings are very helpful to find common fixed point of mappings under any kind of contractive conditions in merger space whether it is rational expression or general.

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